# Relativistic Gamow Vectors I
# Derivation from Poles of the $S$-Matrix

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**Abstract**

A state vector description for relativistic resonances is derived from the first order pole of the $j$-th partial $S$-matrix at the invariant square mass value $\mathsf{s}_R = (m - i\Gamma/2)^2$ in the second sheet of the Riemann energy surface. To associate a ket, called Gamow vector, to the pole, we use the generalized eigenvectors of the four-velocity operators in place of the customary momentum eigenkets of Wigner, and we replace the conventional Hilbert space assumptions for the in- and out-scattering states with the new hypothesis that in- and out-states are described by two different Hardy spaces with complementary analyticity properties. The Gamow vectors have the following properties:
-They are simultaneous generalized eigenvectors of the four velocity operators with real eigenvalues and of the self-adjoint invariant mass operator $M = (P_\mu P^\mu)^{1/2}$ with complex eigenvalue $\sqrt{\mathsf{s}_R}$.
- They have a Breit-Wigner distribution in the invariant square mass variable $\mathsf{s}$ and lead to an exactly exponential law for the decay rates and probabilities.

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# 1 Introduction

This paper is the first of two papers which give the theoretical and mathematical foundations and the detailed derivations of certain results that were used in a preceding paper on the mass and width of relativistic unstable particles [1]. In particular, it gives the construction of relativistic Gamow vectors which were used in [1] to describe the states of a relativistic resonance. This construction is done in complete analogy to the construction of the non-relativistic Gamow vectors in the Rigged Hilbert Space (RHS) formulation of time asymmetric quantum mechanics [2, 3]. We shall briefly review some of the key properties of the non-relativistic theory in the next section. However, the relativistic theory does not require the non-relativistic theory as a backdrop and can be developed starting with the representation spaces of the Poincaré group.

The existence of a fundamental time asymmetry in quantum physics has been noted in the literature, e.g., [4] and [5], and the irreversible character of quantum mechanical decay is fairly well known [6]. However an exact mathematical theory of decay that accommodates this property, especially in the relativistic domain, did not exist. Time asymmetric boundary conditions were incorporated in scattering theory, often unwittingly, by the heuristic Lippmann-Schwinger equations [7] for the in- and out- plane wave states $|E, b^{\pm}\rangle$. These were represented by Dirac kets which were supplementary to an otherwise time symmetric theory in Hilbert space, where the time evolution is necessarily reversible and given by a unitary group. In contrast, the time asymmetric quantum mechanics in Rigged Hilbert Spaces [2, 3] provides a precise definition of the Lippmann-Schwinger Dirac kets $|E, b^{\pm}\rangle$ as functionals in a pair of RHS's of Hardy class and defines the non-relativistic Gamow vectors using the analytic extension of the Lippmann-Schwinger kets into the complex energy half-planes.

The standard relativistic quantum theory is based on the unitary irreducible representations (UIR) of the Poincaré group [8, 9] which describe stable elementary particles. More complicated relativistic quantum systems are described by direct sums (towers) or by direct products (combinations) of UIR's. Within this frame of unitary representations of the relativistic space-time symmetry group, resonances cannot be described as elementary autonomous physical systems.

Our construction of the relativistic Gamow vectors as vectors associated to the $S$-matrix pole at $\mathsf{s} = \mathsf{s}_R$ will lead to irreducible *semigroup* repre-

sentations of causal Poincaré transformations. Like Wigner's unitary group representations $(j, m^2)$ of space-time transformations, these semigroup representations are characterized by two numbers $(j, \mathsf{s}_R)$, the spin $j$ and the invariant mass squared $\mathsf{s}_R = (M_R - i\Gamma_R/2)^2$. The semigroup representations describe relativistic quasistable particles by relativistic Gamow vectors. Further, the transformation properties of these state vectors under Poincaré transformations will show that their time evolution is exponential, and that the coefficient of this exponential time evolution is *precisely* $\gamma\Gamma_R/2$ ($\Gamma_R/2$ in the rest frame) from which we expect that the lifetime is $\tau = \hbar/\Gamma_R$. Using Gamow vectors for the $Z$-boson will thus define its width as $\Gamma_R = -2\text{Im}\sqrt{\mathsf{s}_R}$ and remove the ambiguity in the definition of its mass and width [10, 11].

The parameters $(j, \mathsf{s}_R)$ that characterize the semigroup representations should be related to a definition of the relativistic resonance. We therefore start with the most widely accepted definition of a resonance as the first order pole of the partial $S$-matrix $S_j(\mathsf{s})$ with angular momentum $j$ (perhaps spin-parity $j^\pi$) located at $\mathsf{s} = \mathsf{s}_R$ (in the second sheet). We then analytically extend the relativistic Lippmann-Schwinger kets, $|j, \mathsf{s}, b^\pm\rangle$, from the "physical" values $\mathsf{s}_0 \leq \mathsf{s} < \infty$ into the complex plane and define the relativistic Gamow kets $|[j, \mathsf{s}_R], b^-\rangle$ by the integral around the resonance pole at $\mathsf{s} = \mathsf{s}_R$. These relativistic Gamow kets are generalized eigenvectors of the self-adjoint mass operator $(P_\mu P^\mu)^{1/2}$ with complex eigenvalue $\sqrt{\mathsf{s}_R}$. They provide the state vector description of the physical entity described by the pole term in the relativistic $S$-matrix, i.e., the relativistic Breit-Wigner amplitude.

Phenomenologically one often takes the point of view that resonances or decaying particles are autonomous quantum physical entities characterized by two real values, either the resonance mass $m$ and resonance width $\Gamma$ [11], or the mass $m$ and lifetime $\tau$. One always assumed the relation $\tau = \hbar/\Gamma$, though this could be established only in the Weisskopf-Wigner approximation and even the validity of the exponential law was questioned.

Stability or the value of the lifetime does not appear to be a criterion for elementarity of a relativistic particle. Therefore, a mathematical framework capable of both uniting the notions of Breit-Wigner resonance and exponentially decaying particle and characterizing stable and quasi-stable states on the same footing has a strong theoretical appeal.

Such a characterization has been accomplished for the non-relativistic case [3], where a decaying state has been described by a generalized eigenvector of the (self adjoint, semi-bounded) Hamiltonian with a complex eigenvalue $z_R = E_R - i\Gamma_R/2$ in much the same way as stable particles are described

by eigenvectors of the Hamiltonian with real eigenvalues. The analogous case for the relativistic theory will be discussed in this paper.

# 2 Key Features of Non-relativistic Gamow Vectors

In the standard Hilbert space formulation of quantum mechanics, such vectors with complex eigenvalues do not exist, and for their formulation in the non-relativistic case one had to go to the Rigged Hilbert Space description of quantum mechanics. This was not a revolutionary step since the mathematical definition of Dirac kets already required the Rigged Hilbert Space. The generalized eigenvectors with complex eigenvalues, $\psi^G \equiv |E_R - i\Gamma_R/2^-\rangle\sqrt{2\pi\Gamma}$, which we called Gamow vectors because of their property (2.3) below, were then obtain from the pole term of the $S$-matrix. The real parameters $E_R$ and $\Gamma_R$ are respectively interpreted as resonance energy and resonance width, for reasons that will become clear from their property (2.3) below.

Like Dirac kets, Gamow vectors are functionals of a Rigged Hilbert Space (Appendix A):

$$\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times : \quad \psi^G \in \Phi_+^\times \tag{2.1}$$

and the mathematical meaning of the eigenvalue equation $H^\times|z_R^-\rangle = z_R\,|z_R^-\rangle$ is:

$$\langle H\psi|z_R^-\rangle \equiv \langle\psi|H^\times|z_R^-\rangle = z_R\,\langle\psi|z_R^-\rangle \text{ for all } \psi\in\Phi_+\,. \tag{2.2}$$

The conjugate operator $H^\times$ of the Hamiltonian $H$ is defined, by the first equality in (2.2), as the extension of the Hilbert space adjoint operator $H^\dagger$ of $H$ to the space of functionals $\Phi_+^\times$ [1].

The non-relativistic Gamow vectors have the following properties:

1. They have an asymmetric time evolution and obey an exponential law:

$$\psi^G(t) = e_+^{-iH^\times t}|E_R - i\Gamma/2^-\rangle = e^{-iE_Rt}e^{-\Gamma t/2}|E_R - i\Gamma/2^-\rangle, \quad \textit{only for } t\geq 0\,. \tag{2.3}$$

[1]For (essentially) self-adjoint $H$, $H^\dagger$ is equal to (the closure of) $H$; but we shall use the definition (2.2) also for unitary operator $U$ where $U^\times$ is the extension of $U^\dagger$ and not $U$.

There is another Gamow vector $\tilde{\psi}^G = |E_R + i\Gamma/2^+\rangle \in \Phi_-^\times$, and a semigroup $e_-^{-iH^\times t}$ for $t \leq 0$ in another Rigged Hilbert Space $\Phi_- \subset \mathcal{H} \subset \Phi_-^\times$ (with the same $\mathcal{H}$) with the asymmetric evolution

$$\tilde{\psi}^G = e^{-iH^\times t}|E_R + i\Gamma/2^+\rangle = e^{-iE_R t}e^{\Gamma t/2}|E_R + i\Gamma/2^+\rangle, \qquad \textit{only for } t \leq 0\,. \tag{2.4}$$

2. The $\psi^G$ ($\tilde{\psi}^G$) is derived as a functional associated to the resonance pole term at $z_R = E_R - i\Gamma/2$ (at $z_R^* = E_R + i\Gamma/2$) in the second sheet of the analytically continued $S$-matrix.

3. The Gamow vectors have a Breit-Wigner energy distribution

$$\langle^- E|\psi^G\rangle = i\sqrt{\frac{\Gamma}{2\pi}}\,\frac{1}{E - (E_R - i\Gamma/2)}\,,\ -\infty_{II} < E < \infty\,. \tag{2.5}$$

where $-\infty_{II}$ means that it extends to $-\infty$ on the second sheet of the $S$-matrix (whereas the standard Breit-Wigner terminates at the threshold $E = 0$).

4. The decay probability $P(t) = Tr(\Lambda_\eta|\psi^G(t)\rangle\langle\psi^G(t)|)$ of $\psi^G(t)$, $t \geq 0$, into the final non-interacting decay products of the channel described by $\Lambda_\eta$ can be calculated as a function of time. From this the decay rate $R(t) = \frac{dP(t)}{dt}$ is obtained by differentiation as $R(t) = \Gamma^{(\eta)}e^{-\Gamma t}$ where $\Gamma^{(\eta)}$ is the partial width for the decay channel $\eta$. And it leads to an exact Golden rule for $\Gamma^{(\eta)}$ which in the Born approximation ($\psi^G \to f^D$, an eigenvector of $H_0 = H - V$; $\Gamma/E_R \to 0$; $E_R \to E_0$) goes into Fermi's Golden rule No.2 of Dirac. As a consequence of the exponential law for $R(t)$, the lifetime of the state described by $\psi^G(t)$ is given precisely by $\tau = \hbar/\Gamma$.

We want to generalize these non-relativistic Gamow vectors to the relativistic case and construct representations of the Poincaré group $\mathcal{P}$ that describe relativistic resonances and decaying states very much in the same way as unitary irreducible representations (UIR) of $\mathcal{P}$ describe stable relativistic particles. In order to obtain a state vector description for unstable particles, one has to start from the space of decay products. For a relativistic unstable particle decaying into a two-particle system with masses $m_1$, $m_2$, and

spins $s_1$, $s_2$, the (asymptotically free) decay product space is the direct product space of irreducible representation spaces of the two particles $\mathcal{H}(m_i, s_i)$, $i = 1,\ 2$: $\mathcal{H}_{12} = \mathcal{H}(m_1, s_1) \otimes \mathcal{H}(m_2, s_2)$. A decaying state vector can then be obtained, in analogy to the non-relativistic case, by extending the basis vectors of $\mathcal{H}_{12}$ to complex energy, in a manner which is consistent with the dynamical description of unstable particles as poles of the analytically continued $S$-matrix. Since resonances occur in one-particular partial-wave $j_R^\pi$, (which designates the spin (total angular momentum in the rest frame) and parity of the unstable particle), we need to use the basis vectors of $\mathcal{H}_{12}$ which are diagonal in total angular momentum in order to obtain the relativistic Gamow vectors. These angular momentum basis vectors are obtained by the reduction of the direct product $\mathcal{H}(m_1, s_1) \otimes \mathcal{H}(m_2, s_2)$ into a continuous direct sum of irreducible representation spaces

$$\mathcal{H}_{12} = \sum_{j\eta} \int_{(m_1+m_2)^2}^{\infty} d\mu(\mathsf{s}) \mathcal{H}_n^\eta(\mathsf{s}, j)\,, \tag{2.6}$$

where $\mathsf{s}$ is the invariant mass square (Mandelstam parameter) for the two-particle system $\mathsf{s} = (p_1 + p_2)^2$, and $\eta$ and $n$ are degeneracy and particle species labels respectively [12, 13]. As discussed in Sections 3.2 and 4 below, the partial $S$-matrix is a function of $\mathsf{s}$ which is analytic except for poles and branch cuts. The complex poles at $\mathsf{s} = \mathsf{s}_R$ are the ones associated with the unstable particles, and to these values the analytic extension in the $\mathsf{s}$-variable will be performed.

The extension of the invariant mass square $\mathsf{s}$ to a complex value leads necessarily to complex momenta, since $\mathsf{s} = E^2 - \boldsymbol{p}^2$. As Lorentz transformations intermingle energy and momenta, this in general leads to complex momentum representations of the Poincaré group. To obtain a description of unstable particles with minimal modifications to the stable particle case, and to restrict the set of complicated complex momentum representations of $\mathcal{P}$ [14], we will consider representations of $\mathcal{P}$ where complexness is due only to the complexness of the mass square $\mathsf{s}$. These representations are "minimally complex", in the sense that, while the invariant mass squared $\mathsf{s}$ is complex, the 4-velocities $\hat{p}^\mu = p^\mu/\sqrt{\mathsf{s}}$ remain real. This construction was motivated by a remark by D. Zwanziger [15] and is based on the fact that the velocity kets provide as valid a basis for the UIR's of the Poincaré group as Wigner's momentum kets. Moreover, the 4-velocity eigenvectors are often more useful for physical reasoning, because 4-velocities seem to fulfill to a rather good

approximation "velocity super-selection rules" which the momenta do not [16].

The reduction of $\mathcal{H}_{12}$ in the velocity-angular momentum basis $|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]\eta, n\rangle$ was performed in [13] as a preparation for this paper. The discussion there extends to an unstable particle decaying into more than two particles, but the reduction of the decay product space into a direct sum of UIR spaces becomes much more tedious than for the two-particle case. Hence, in general, with the use of velocity-angular momentum kets, the relativistic Gamow vectors will be defined not as momentum eigenvectors but as 4-velocity eigenvectors in the direct product space of UIR's for the decay products of the resonance $R$. This definition requires the extension of the invariant square mass $\mathsf{s} = (p_1 + p_2 + \cdots)^2$, where $p_1$, $p_2 \cdots$ are the momenta of the decay products of $R$, into the complex values. In analogy to the non-relativistic case, we define the resonances $R$ by poles of the analytically continued relativistic partial $S$-matrix with total angular momentum $j = j_R$ and pole position $\mathsf{s} = \mathsf{s}_R = (M_R - i\Gamma_R/2)^2$. As discussed in Section 3.1 below, there appear to be phenomenological and theoretical arguments in favor of this definition. This means that the relativistic particle is characterized by the value of its spin $j_R$ and its mass $w_R \equiv \sqrt{\mathsf{s}_R} = M_R - i\Gamma_R/2$, where $\Gamma_R$ is zero for stable particles, but $w_R$ is complex if the particle is unstable. Just as the mass and spin of a stable particle permit its association to an irreducible unitary representation of the Poincaré group characterized by these values, it will be shown in a forthcoming paper that the complex mass $w_R$ and spin $j_R$ concatenate an unstable particle to an irreducible representation of the Poincaré group, only these irreducible representations are not unitary. In fact, it turns out that they are irreducible representations of the causal Poincaré *semigroup*, defined as the semi-direct product of the group of homogeneous Lorentz transformations with the *semigroup* of space-time translations into the forward light cone. The fundamental mathematical object needed in constructing these semigroup representations is the relativistic Gamow vector. The main technical result of this paper is its derivation, while the discussion of the semigroup representations is deferred to a later paper.

# 3 Relativistic Gamow Vectors From Poles of the Relativistic $S$-matrix

## 3.1 Introduction

As discussed above, our aim is to obtain the relativistic Gamow vectors from the pole term of the relativistic $S$-matrix in complete analogy to the way the non-relativistic Gamow vectors were obtained [17]. For the sake of definiteness, we discuss here the case of resonance formation in an elastic scattering process $a\,b \to R \to a\,b$ for which one may consider $\pi^+\pi^- \to \rho \to \pi^+\pi^-$ [18] or $e^+e^- \to Z \to e^+e^-$ [10] as examples.

In the absence of a vector space description of a resonance, we shall also in the relativistic theory define the unstable particle by the pole of the analytically continued partial $S$-matrix with angular momentum $j = j_R$ at the value $\mathsf{s} = \mathsf{s}_R \equiv (M_R - \frac{i}{2}\Gamma_R)^2$ of the invariant mass square variable (Mandelstam variable) $\mathsf{s}$ [19]. This means that the mass $M_R$, width $\Gamma_R$ (which will be shown to be connected to the lifetime by $\tau_R = \hbar/\Gamma_R$), and spin $j_R$ are the intrinsic properties that define a quasistable relativistic particle [2]. As will be shown below in Section 5, the $j$-th partial $S$-matrix can be separated into a background term and a pole term which is given by the relativistic Breit-Wigner amplitude

$$a_{j_R} = \frac{f_n \Gamma_R/2}{\mathsf{s} - (M_R - \frac{i}{2}\Gamma_R)^2} \quad \text{with } -\infty_{II} \leq \mathsf{s} \leq \infty\,, \tag{3.1}$$

where the negative values of $\mathsf{s}$ are on the second sheet.

[2]The more conventional definition, used in [11], of resonance mass and width is $M_\rho$, $\Gamma_\rho$, which are defined in terms of $M_R$, $\Gamma_R$ by

$$\mathsf{s}_R \equiv M_\rho^2 - iM_\rho\Gamma_\rho = M_R^2\left(1 - \frac{1}{4}\left(\frac{\Gamma_R}{M_R}\right)^2\right) - iM_R\Gamma_R \tag{3.1b}$$

or $M_\rho = M_R\sqrt{1 - \frac{1}{4}\left(\frac{\Gamma_R}{M_R}\right)^2}$ and $\Gamma_\rho = \Gamma_R\left(1 - \frac{1}{4}\left(\frac{\Gamma_R}{M_R}\right)^2\right)^{-1/2}$. $M_\rho^2$ is the peak position of the relativistic Breit-Wigner probability $|a_{j_3}(\mathsf{s})|^2$. The exponential time evolution of the Gamow vectors which will be derived in the forthcoming sequel to this paper, shows that $\Gamma_R$, and not $\Gamma_\rho$, is the inverse lifetime. Since only $\Gamma_R$ (not $\Gamma_Z$ or $\Gamma_\rho$) of the parameterization (3.1) precisely fulfills the identity $\Gamma_R = \hbar/\tau$, we call $\Gamma_R$ the "width" of the relativistic resonance. For the $\rho$-meson $\left(\frac{\Gamma_R}{M_R}\right)^2 \approx 0.03$ and for the $Z$-boson $\left(\frac{\Gamma_R}{M_R}\right)^2 \approx 7 \times 10^{-4}$. Usually the difference between $M_\rho$ and $M_R$ is an order of magnitude smaller and/or well within the experimental error for the experimental mass; however for the $\rho$-meson and in particular for the $Z$-boson this difference is significant, see [20].

However we want to mention that the $S$-matrix pole definition with the parameterization by $M_R$, $\Gamma_R$ is neither the only nor the universally accepted definition of the parameters mass $m$ and width $\Gamma$ of a relativistic resonance. The particle data table uses $M_\rho$ and $\Gamma_\rho$ as mass and width [2]. And for precision data as in the line shape analysis for the $Z$-boson [10], one prefers an energy dependent width replacing $M_\rho \Gamma_\rho \to \sqrt{\mathsf{s}}\,\Gamma(\mathsf{s})$, cf. equation (35.53) of [11]. This gives still another parameterization which in the case of the $Z$-boson is called $M_Z$, $\Gamma_Z$ [10]. In the line-shape analysis for the $Z$-boson, these three parameterizations lead to three different values for the $Z$-boson mass and width [20] which fit the experimental data equally well. Therefore the line-shape data do not provide a phenomenological clue as to how the fundamental parameters $m$, $\Gamma$ of the $Z$-boson should be defined [20]. In the case of the $\rho$-meson, the definition of the resonance parameters given by (3.1) leads to better, process-independent fits of the parameters [21]. Thus, there seems to be a slight phenomenological preference for the $S$-matrix pole definition (3.1) of a relativistic resonance. The parameterizations in terms of $M_R$, $\Gamma_R$ or $M_\rho$ $\Gamma_\rho$ of the $S$-matrix pole position $\mathsf{s}_R$ are phenomenologically equivalent. One of the results of our theory will be that only $\Gamma_R$ and not $\Gamma_\rho$ (or $\Gamma_Z$) will be connected to the lifetime by $\Gamma_R = \hbar/\tau_R$, which is the theoretical argument in favor of the parameterization $\mathsf{s}_R = (M_R - i\Gamma_R/2)^2$.

In order to make the analytic continuation in the partial $S$-matrix with angular momentum $j$, we need the angular momentum basis vectors for the scattering states. Thus we replace the non-relativistic angular momentum basis vectors in the derivation for the non-relativistic Gamow vectors with:

$$|\,E\,\rangle \equiv |\,E l l_3 \eta\,\rangle \to |\,\hat{\boldsymbol{p}} j_3 [w = \sqrt{\mathsf{s}}, j]\,\rangle \tag{3.2}$$

$$|\,E^\pm\,\rangle \equiv |\,E l l_3 \eta^\pm\,\rangle \to |\,\hat{\boldsymbol{p}} j_3 [w = \sqrt{\mathsf{s}}, j]^\pm\,\rangle\,. \tag{3.2$^\pm$}$$

The two sets of bases are related by the Moeller wave operators

$$|\,\hat{\boldsymbol{p}} j_3 [w = \sqrt{\mathsf{s}}, j]^\pm\,\rangle = \Omega^\pm |\,\hat{\boldsymbol{p}} j_3 [w = \sqrt{\mathsf{s}}, j]\,\rangle\,.$$

We shall analytically extend the Dirac kets (3.2$^\pm$) in the variable $\mathsf{s}$ from the physical values to the complex values on the Riemann surface of the $S$-matrix. The analytic extension of $|\hat{\boldsymbol{p}}, j_3, [\mathsf{s}, j]^-\rangle$ at the pole position $\mathsf{s}_R$ yields the relativistic Gamow vector.

## 3.2 The $S$-matrix [3]

In a scattering experiment, the experimentalist prepares an initial state $\phi^{in}$ [4], describing the non-interacting projectile and target beams, at $t \to -\infty$. Then, later, at $t' \to +\infty$ the experimentalist measures or registers an observable [4] $|\psi^{out}\rangle\langle\psi^{out}|$. It is assumed that the time translation generator $H$ can be divided into two terms, the "free-particle" Hamiltonian $K(= P_1^0 + P_2^0$ at rest) and an interaction part $V$:

$$H = K + V\,,$$

where the split of $H$ into $K$ and $V$ will be different if different in- and out-particles are involved.

The state vectors $\phi^{in}(t) = e^{-iKt}\phi^{in}$ and the observable vectors $\psi^{out}(t') = e^{-iKt'}\psi^{out}$ evolve in time according to the free Hamiltonian $K$. When the beams reach the interaction region, the free in-state vector $\phi^{in}$ turns into the exact state vector $\phi^+$ whose time evolution is given by the exact Hamiltonian $H = K + V$:

$$\Omega^+\phi^{in}(t) \equiv \phi^+(t) = e^{-iHt}\phi^+ = \Omega^-\phi^{out}(t)\,. \tag{3.3}$$

Here $t$ is the proper time in the center-of-mass of the projectile and target. This vector $\phi^+$ leaves the interaction region and becomes the well determined state $\phi^{out}$. The state vector $\phi^{out}$ is determined from $\phi^{in}$ by the dynamics of the scattering process:

$$\phi^{out} = S\phi^{in}, \quad S = \Omega^{-\dagger}\Omega^+ \tag{3.4}$$

The state $\phi^{in}$ and thus $\phi^+$ and also $\phi^{out}$ are determined by the preparation apparatus (the accelerator).

A scattering experiment consists of a preparation apparatus and a registration apparatus (detector).

[3] Here we closely follow chapter 3 of [9] in order to both display the analogy (and comparability) and expose the differences between our development and the standard views in relativistic quantum theory. Our notation transcribes into that of [9] as $\{\phi^{in/out}, \psi^{out}\} \to \Phi_g$ and $\{\phi^+, \psi^-\} \to \Psi_g^\pm$. In [9], the multi-particle basis vectors are also denoted by $\Psi_\alpha^\pm$ where $\alpha = \{p_1\sigma_1 n_1, p_2\sigma_2 n_2, \cdots\}$, $\sigma$ is the third component of the spin, and $n$ is the species label.

[4] In realistic experiments the states are not pure but mixtures $W^{in} = \sum w_\alpha |\phi_\alpha^{in}\rangle\langle\phi_\alpha^{in}|$ and the observables are not given by projection operators but by $\Lambda^{out} = \sum \lambda(\beta)|\psi_\beta^{out}\rangle\langle\psi_\beta^{out}|$

The registration apparatus registers an observable $|\psi^{out}\rangle\langle\psi^{out}|$ outside the interaction region. This observable vector $\psi^{out}$ comes from a vector

$$\psi^{-} = \Omega^{-}\psi^{out} \tag{3.5}$$

in the interaction region. The observable $\psi^{out}$ is of course not the same as the state $\phi^{out}$, since $\phi^{out}$, and thus $\phi^{+}$ and $\phi^{in}$, is defined by the accelerator and $\psi^{out}$, and thus $\psi^{-}$, is defined by the detector. Hence he set of vectors $\psi^{-}$ may be, and in our case is, distinct from the set of vectors $\phi^{+}$.

The probability amplitude to register the observable $|\psi^{out}\rangle\langle\psi^{out}|$ in the state $\phi^{out}$, which–by simple calculations using (3.3) and (3.4)–is the same as the probability amplitude for the observable $|\psi^{-}\rangle\langle\psi^{-}|$ in the state $\phi^{+}$, is the scalar product

$$(\psi^{out}, \phi^{out}) = (\psi^{out}, S\phi^{in}) = (\Omega^{-}\psi^{out}, \Omega^{+}\phi^{in}) = (\psi^{-}, \phi^{+})\,.$$

The fundamental probabilities of quantum theory (Born probability) are the probabilities for the observables $|\psi^{-}\rangle\langle\psi^{-}|$ in the state $|\phi^{+}\rangle\langle\phi^{+}|$ (or for the observable vector $\psi^{-}$ in the state vector $\phi^{+}$) given by

$$Tr(|\psi^{-}\rangle\langle\psi^{-} \mid \phi^{+}\rangle\langle\phi^{+}|) = |(\psi^{-}, \phi^{+})|^{2}\,.$$

The vectors $\phi^{+}_{\alpha}$ are called the in-states and the vectors $\psi^{-}_{\beta}$ are called the out-states and the array of complex amplitudes $(\psi^{-}_{\beta}, \phi^{+}_{\alpha})$ is called the $S$-matrix. The labels $\alpha$ and $\beta$ stand for a whole collection of *discrete* quantum numbers. The $S$-matrix is also defined when $\alpha$ and/or $\beta$ are continuous labels (of basis vectors for the space of the $\{\psi^{-}\}$ and the space of the $\{\phi^{+}\}$) only then the $S$-matrix does not represent probability but a probability density. "It should be stressed that in-states $\phi^{+}$ and out-states $\psi^{-}$ do not inhabit two different Hilbert spaces" [9] but they do inhabit two *different* (dense) subspaces of the same Hilbert space $\mathcal{H}$; these two dense subspaces we call $\Phi_{-} \equiv \{\phi^{+}\}$ and $\Phi_{+} \equiv \{\psi^{-}\}$. This is the only new hypothesis by which our time asymmetric quantum theory in Rigged Hilbert Space differs from the standard Hilbert space quantum theory. Thus the Rigged Hilbert Space theory distinguishes meticulously between prepared states (in-states) $\{\phi^{+}\}$ and observables (out-states) $\{\psi^{-}\}$ by means of two different Rigged Hilbert Spaces of Hardy class:

$$\begin{array}{ll} \phi^{+} \in \Phi_{-} \subset \mathcal{H} \subset \Phi^{\times}_{-} & \text{for the prepared in-states defined by the} \\ & \text{preparation apparatus (accelerator)} \\ \psi^{-} \in \Phi_{+} \subset \mathcal{H} \subset \Phi^{\times}_{+} & \text{for the registered out-states (observables)} \\ & \text{defined by the detector} \end{array} \tag{3.6}$$

In contrast, the conventional scattering theory assumes $\Phi_- = \Phi_+ (= \mathcal{H})$. As a consequence of this new hypothesis, an arbitrary in-state $\phi^+ \in \Phi_-$ *cannot* be expanded as a sum of out-states $\psi_\alpha^- \in \Phi_+$. However (for instance in the non-relativistic theory), any in-state $\phi^+$ can be expanded with respect to a generalized basis system $|E^-\rangle \in \Phi_+^\times$ of the out-states $\psi^-$ ($\psi^- = \int dE|E^-\rangle\langle^- E\,|\,\psi^-\rangle$) using the $S$-matrix :

$$\phi^+ = \int_0^\infty dE|E^-\rangle\langle^- E\,|\,\phi^+\rangle = \int_0^\infty dE|E^-\rangle S(E+i0)\langle^+ E\,|\,\phi^+\rangle\,.$$

In the non-relativistic theory the boundary conditions required of the spaces $\Phi_-$ and $\Phi_+$ lead to distinct analyticity properties for the energy wave functions $\langle^+ E\,|\,\phi^+\rangle$ of $\phi^+ \in \Phi_-$ and the energy wave functions $\langle^- E\,|\,\psi^-\rangle$ of $\psi^- \in \Phi_+$ [22]. To wit

$$\psi^- \in \Phi_+ \text{ if and only if } \langle^- E|\psi^-\rangle = \langle E|\psi^{out}\rangle \in \mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}_+} \tag{3.7a}$$

$$\phi^+ \in \Phi_- \text{ if and only if } \langle^+ E|\phi^+\rangle = \langle E|\phi^{in}\rangle \in \mathcal{S} \cap \mathcal{H}_-^2|_{\mathbb{R}_+} \tag{3.7b}$$

where: $\mathcal{S}$ denotes the Schwartz space; $\mathcal{H}_\pm^2$, the Hardy class functions (Appendix B) which are boundary values of functions that are analytic in the open $\begin{matrix}\text{upper}\\\text{lower}\end{matrix}$ half plane of the complex energy plane; and $|_{\mathbb{R}_+}$, the restriction to the physical values of the energy, $\mathbb{R}_+ = [0, \infty)$. As the lower half complex energy plane we take the second Riemann sheet of the $S$-matrix reached from the physical values $\langle^+ E + i0\,|\,\phi^+\rangle$, $E \in \mathbb{R}_+$ by burrowing down through the cut along the positive real axis $\mathbb{R}_+$. As the upper half complex energy plane we take the second Riemann sheet of the $S$-matrix reached from the values $\langle^- E - i0\,|\,\psi^-\rangle$, $E \in \mathbb{R}_+$ on the physical sheet by burrowing up through the cut along $\mathbb{R}_+$. In the non-relativistic case (3.7b) and (3.7a) could be obtained from a mathematical formulation of causality (using the Paley-Wiener theorem) [22]. An important consequence of the distinction between $\Phi_+$ and $\Phi_-$ obtained in (3.7) is that the time evolution operator $U(t) = e^{iHt}$, which is represented by a group on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+)$, splits into two semigroups when restricted to $\Phi_\pm$ [17]. This means that $U(t)|_{\Phi_+}$ is a continuous operator on $\Phi_+$ only for $t \geq 0$ and $U(t)|_{\Phi_-}$ is a continuous operator on $\Phi_-$ only for $t \leq 0$. It is this semigroup property that is at the origin of the time asymmetric evolution equations (2.3) and (2.4).

In the relativistic theory, in particular for the elastic scattering of two particles, the Hilbert space $\mathcal{H}$ of (3.6) is the direct integral (2.6) of irreducible

representation spaces $\mathcal{H}_n^\eta(\mathsf{s}, j)$ labeled by the Mandelstam variable $\mathsf{s} = (p_1 + p_2)^2$ and the total angular momentum $j$ of the two particles. For the measure $d\mu(\mathsf{s})$ in (2.6) we take the Lebesgue measure $d\mathsf{s}$. In what follows, we will restrict our discussion to the elastic scattering of two spinless particles with the same mass $m$. We use as a basis for $\Phi_\pm$ the angular momentum velocity basis vectors, $\{|\hat{\boldsymbol{p}} j_3[\mathsf{s}j]^\mp\rangle\}$ [13]. With these basis vectors, the Dirac basis vector expansion for an in-state (of two equal mass spinless particles) $\phi^+ \in \Phi_-$ is:

$$\phi^+ = \sum_{jj_3} \int_{4m^2}^{\infty} d\mathsf{s} \int \frac{d^3\hat{p}}{2\hat{p}^0} |\hat{\boldsymbol{p}} j_3[\mathsf{s}j]^+\rangle \langle \hat{\boldsymbol{p}} j_3[\mathsf{s}j]^+|\phi^+\rangle \,, \tag{3.8}$$

and for an out-observable $\psi^- \in \Phi_+$ is:

$$\psi^- = \sum_{jj_3} \int_{4m^2}^{\infty} d\mathsf{s} \int \frac{d^3\hat{p}}{2\hat{p}^0} |\hat{\boldsymbol{p}} j_3[\mathsf{s}j]^-\rangle \langle \hat{\boldsymbol{p}} j_3[\mathsf{s}j]^-|\psi^-\rangle \,. \tag{3.9}$$

In (3.8) and (3.9), $\langle \hat{\boldsymbol{p}} j_3[\mathsf{s}j]^+|\phi^+\rangle$ and $\langle \hat{\boldsymbol{p}} j_3[\mathsf{s}j]^-|\psi^-\rangle$ are the wave functions of $\phi^+$ and $\psi^-$ along the velocity basis vectors. We shall make the hypothesis that these wave functions have the same analyticity properties in the invariant mass squared $\mathsf{s} = (E^{\mathrm{cm}})^2$ as the energy wave functions in (3.7). However, in the relativistic case, due to the mathematical requirement of the invariance of the subspaces $\Phi_\mp$ under the action of the generators of the Poincaré group, a closed subspace $\tilde{\mathcal{S}}$ of the Schwartz space $\mathcal{S}$ of (3.7) has to be considered. The subspace $\tilde{\mathcal{S}}$, constructed in [23], is the space of Schwartz functions which vanish at zero faster than any polynomial. This requirement also assures that the zero mass states do not contribute to the Gamow vector (see (5.17)). This avoids the difficulty that the four velocity operators, which is centrally significant to our construction of Gamow vectors, cannot be meaningfully defined in the zero-mass case in any obvious way. The features of the space $\tilde{\mathcal{S}}$ which are needed for the construction of the relativistic Gamow vectors are as follows [23]:

**Proposition 3.1.** The triplets

$$\tilde{\mathcal{S}} \cap \mathcal{H}^2_\mp|_{\mathbb{R}_{\mathsf{s}_0}} \subset L^2(\mathbb{R}_{\mathsf{s}_0}) \subset \left(\tilde{\mathcal{S}} \cap \mathcal{H}^2_\mp|_{\mathbb{R}_{\mathsf{s}_0}}\right)^\times \tag{3.10}$$

form a pair of Rigged Hilbert Spaces.

In (3.10), $\mathbb{R}_{\mathsf{s}_0}$ is the set of physical values of the Mandelstam variable $\mathsf{s}$ for the scattering process, $\mathbb{R}_{\mathsf{s}_0} = [(m_1+m_2)^2, \infty)$.

**Proposition 3.2.** The space $\tilde{\mathcal{S}}$ is endowed with a nuclear Fréchet topology such that multiplication by $\mathsf{s}^{\frac{n}{2}}$,

$$\mathsf{s}^{\frac{n}{2}} \,:\, \tilde{\mathcal{S}} \cap \mathcal{H}^2_{\pm} \to \tilde{\mathcal{S}} \cap \mathcal{H}^2_{\pm}\,, \quad n = 1, 2, 3, \cdots$$

is a continuous linear operator in the topology of $\tilde{\mathcal{S}}$ .

Thus the relativistic characterization of $\Phi_\pm$ analogous to (3.7a) and (3.7b) is:

$$\psi^- \in \Phi_+ \quad \text{if and only if } \langle \hat{\boldsymbol{p}} j_3 [\mathsf{s}j]^- | \psi^- \rangle \in \tilde{\mathcal{S}} \cap \mathcal{H}^2_+|_{\mathbb{R}_{\mathsf{s}_0}} \times \mathcal{S}(\mathbb{R}^3) \tag{3.11a}$$

$$\phi^+ \in \Phi_- \quad \text{if and only if } \langle \hat{\boldsymbol{p}} j_3 [\mathsf{s}j]^+ | \phi^+ \rangle \in \tilde{\mathcal{S}} \cap \mathcal{H}^2_-|_{\mathbb{R}_{\mathsf{s}_0}} \times \mathcal{S}(\mathbb{R}^3)\,; \tag{3.11b}$$

where $\mathbb{R}^3$ is the space of components of the 4-velocity and the Hilbert space $\mathcal{H}$ of (3.6) is realized by the function space

$$L^2(\mathbb{R}_{\mathsf{s}_0}, d\mathsf{s}) \times L^2 \left( \mathbb{R}^3, \frac{d^3\hat{p}}{2\hat{p}^0} \right) \,. \tag{3.12}$$

In (3.11) as in (3.7), $\mathcal{H}^2_+$ means the functions of Hardy class analytic in the upper half of the second sheet of the $\mathsf{s}$-plane and $\mathcal{H}^2_-$ means the functions of Hardy class analytic in its lower half. Specifically, the physical values $\langle {}^+\mathsf{s} - i0 \,|\, \phi^+ \rangle$ are the boundary values of functions analytic in the lower half of the second sheet and the $\langle {}^-\mathsf{s} + i0 \,|\, \psi^- \rangle$ are the boundary values of functions analytic in the upper half of the second sheet. These analyticity properties on the second sheet of the complex $\mathsf{s}$-Riemann surface will turn out to be important because the (pairs of) resonance poles of the $S$-matrix are located on the second Riemann sheet.

By virtue of Proposition 3.2, the total momentum operators $P_\mu = P_{1\mu} + P_{2\mu}$ and the invariant mass square operator $M^2 = P_\mu P^\mu$ are $\tau_{\Phi_\pm}$-continuous operators; hence their conjugates [5], $P^\times_\mu$ and $M^{2^\times}$, are well defined on $\Phi^\times_\pm$. This can be seen by considering the realization, for instance, of the vectors $P_\mu \psi^-$ and $M^2\psi^-$:

$$\langle P_\mu \psi^- | \hat{\boldsymbol{p}} j_3 [\mathsf{s}j]^- \rangle = \langle \psi^- | P^\times_\mu | \hat{\boldsymbol{p}} j_3 [\mathsf{s}j]^- \rangle = \sqrt{\mathsf{s}} \hat{p}_\mu \langle \psi^- | \hat{\boldsymbol{p}} j_3 [\mathsf{s}j]^- \rangle \,, \tag{3.13a}$$

$$\langle M^2 \psi^- | \hat{\boldsymbol{p}} j_3 [\mathsf{s}j]^- \rangle = \langle \psi^- | M^{2^\times} | \hat{\boldsymbol{p}} j_3 [\mathsf{s}j]^- \rangle = \mathsf{s} \langle \psi^- | \hat{\boldsymbol{p}} j_3 [\mathsf{s}j]^- \rangle \,. \tag{3.13b}$$

[5] defined by the first equality in (3.13)

According to Proposition 3.2 and the definition of the wave functions $\langle\psi^-|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^-\rangle$ given in (3.11a), the multiplication operators by $\sqrt{\mathsf{s}}\hat{p}_\mu$ and $\mathsf{s}$ which appear in the right hand side of (3.13a) and (3.13b) are $\tau_{\Phi_+}$-continuous. Consequently, $P_\mu$ and $M^2$ are $\tau_{\Phi_+}$-continuous operators, and the conjugate operators $M^{2^\times}$ and $P_\mu^\times$ that appear in (3.13) are everywhere defined, weak*-continuous operators on $\Phi_+^\times$. Hence, (3.13a) and (3.13b) define the functionals $|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^-\rangle$ as generalized eigenvectors of $P_\mu$ and $M^2$. The same discussion applies for the space $\Phi_-$. Summarizing

$$P_\mu\ :\ \Phi_\pm \to \Phi_\pm \quad \text{is}\ \ \tau_{\Phi_\pm}\text{-continuous} \tag{3.14}$$

and

$$P_\mu^\times|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^\mp\rangle = \sqrt{\mathsf{s}}\hat{p}_\mu|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^\mp\rangle\,, \tag{3.15a}$$

$$M^{2^\times}|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^\mp\rangle = \mathsf{s}|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^\mp\rangle\,. \tag{3.15b}$$

We can re-express the generalized eigenvalues of the momentum operator in terms of the three velocity $\boldsymbol{v}$ by noting that $\hat{\boldsymbol{p}} = \gamma\boldsymbol{v} = \frac{\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2}}$, and $1+\hat{\boldsymbol{p}}^2 = \frac{1}{1-\boldsymbol{v}^2} = \gamma^2$. Hence, the eigenvalues in (3.15a) can be rewritten as

$$\begin{aligned} H^\times|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^\mp\rangle &= \gamma\sqrt{\mathsf{s}}|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^\mp\rangle\,,\\ \boldsymbol{P}^\times|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^\mp\rangle &= \gamma\sqrt{\mathsf{s}}\boldsymbol{v}|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^\mp\rangle\,. \end{aligned} \tag{3.16}$$

For the branch of $\sqrt{\mathsf{s}}$ in (3.13), (3.15) and (3.16), we choose

$$-\pi \le \operatorname{Arg}\mathsf{s} < \pi\,. \tag{3.17}$$

This choice of branch, even though irrelevant for the physical values of $\mathsf{s}$, will be needed since we will analytically continue the kets $|\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^\mp\rangle$ to the unphysical second Riemann sheet as described in Section 5.

We shall now consider the $S$-matrix element

$$\begin{aligned} (\psi^{out}, S\phi^{in}) &= (\psi^-, \phi^+)\\ &= \sum_{jj_3}\int\frac{d^3\hat{p}}{2\hat{E}}d\mathsf{s}\sum_{j'j'_3}\int\frac{d^3\hat{p}'}{2\hat{E}'}d\mathsf{s}'\langle\,\psi^-\,|\,\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^-\rangle\\ &\qquad\langle\,\hat{\boldsymbol{p}}j_3[\mathsf{s}j]\,|\,S\,|\,\hat{\boldsymbol{p}}'j'_3[\mathsf{s}'j']\,\rangle\langle^+\hat{\boldsymbol{p}}'j'_3[\mathsf{s}'j']\,|\,\phi^+\,\rangle \end{aligned} \tag{3.18}$$

where we insert into $(\psi^-, \phi^+)$ a complete system of basis vectors [6] and use

$$\begin{aligned}\langle^- \hat{\boldsymbol{p}} j_3[\mathsf{s}j] \,|\, \hat{\boldsymbol{p}}' j_3'[\mathsf{s}'j']^+\rangle &= (\, \Omega^- |\, \hat{\boldsymbol{p}} j_3[\mathsf{s}j]\,\rangle, \Omega^+ |\, \hat{\boldsymbol{p}}' j_3'[\mathsf{s}'j']\,\rangle\,) \\ &= \langle\, \hat{\boldsymbol{p}} j_3[\mathsf{s}j] \,|\, \Omega^{-\dagger}\Omega^+ \,|\, \hat{\boldsymbol{p}}' j_3'[\mathsf{s}'j']\,\rangle \\ &= \langle\, \hat{\boldsymbol{p}} j_3[\mathsf{s}j] \,|\, S \,|\, \hat{\boldsymbol{p}}' j_3'[\mathsf{s}'j']\,\rangle\end{aligned} \tag{3.19}$$

Using the invariance of the $S$ operator with respect to space time translations

$$[P_\mu, S] = 0 \tag{3.20}$$

we obtain

$$\begin{aligned}&\langle\, \hat{\boldsymbol{p}} j_3[\mathsf{s}j]\eta \,|\, S \,|\, \hat{\boldsymbol{p}}' j_3'[\mathsf{s}'j']\eta'\,\rangle \\ &\qquad = \delta(\boldsymbol{p} - \boldsymbol{p}')\delta(p_0 - p_0')\langle\!\langle\, \hat{\boldsymbol{p}} j_3[\mathsf{s}j]\eta \,|\, \tilde{S} \,|\, \hat{\boldsymbol{p}}' j_3'[\mathsf{s}'j']\eta'\,\rangle\!\rangle\end{aligned} \tag{3.21}$$

where $\langle\!\langle \quad |\tilde{S}| \quad \rangle\!\rangle$ is a reduced $S$–matrix element. This we can also write as

$$\begin{aligned}\langle\, \hat{\boldsymbol{p}} j_3[\mathsf{s}j]\eta \,|\, S \,|\, \hat{\boldsymbol{p}}' j_3'[\mathsf{s}'j']\eta'\,\rangle &= 2\hat{E}(\hat{p})\delta(\hat{\boldsymbol{p}} - \hat{\boldsymbol{p}}')\delta(\mathsf{s} - \mathsf{s}') \\ &\qquad \langle\!\langle\, \hat{\boldsymbol{p}} j_3[\mathsf{s}j]\eta \,|\, S \,|\, \hat{\boldsymbol{p}}' j_3'[\mathsf{s}'j']\eta'\,\rangle\!\rangle\end{aligned} \tag{3.22}$$

where $\langle\!\langle \quad |\, S \,| \quad \rangle\!\rangle$ is another reduced matrix element defined by (3.22). In (3.21) and (3.22) we include explicitly the degeneracy quantum number $\eta$ for purposes of clarity and completion, but we will omit it below for the sake of notational convenience. The form (3.22) follows from (3.21) by the defining identities $\hat{\boldsymbol{p}} = \frac{\boldsymbol{p}}{\sqrt{\mathsf{s}}}$, $\hat{p}^0 = \frac{p^0}{\sqrt{\mathsf{s}}}$. From the invariance of the $S$-operator with respect to Lorentz transformations, in particular from $U^\dagger(L^{-1}(\hat{p}))SU(L^{-1}(\hat{p})) = S$ [7] it follows that the reduced matrix element is independent of $\hat{p}$ (i.e., it is the same for all $\hat{p}$ as in the center of mass frame $\hat{\boldsymbol{p}} = \boldsymbol{0}$). Invariance with respect to rotations in the center of mass frame shows then by analogous arguments for the discrete quantum numbers $j_3$ and $j$ that the reduced matrix element is proportional to $\delta_{j_3 j_3'}\delta_{jj'}$ and independent of $j_3$. Since Poincaré transformations do not change the Poincaré invariants $\mathsf{s}$ and $j$, the reduced matrix element can still depend upon $\mathsf{s}$ and $j$. Thus we have

$$\begin{aligned}\langle\, \hat{\boldsymbol{p}} j_3[\mathsf{s}j]\eta \,|\, S \,|\, \hat{\boldsymbol{p}}' j_3'[\mathsf{s}'j']\eta'\,\rangle &= 2\hat{E}(\hat{p})\delta(\hat{\boldsymbol{p}} - \hat{\boldsymbol{p}}')\delta(\mathsf{s} - \mathsf{s}')\delta_{j_3 j_3'}\delta_{jj'} \\ &\qquad \langle\, \eta \,\|\, S_j(\mathsf{s}) \,\|\, \eta'\,\rangle\end{aligned} \tag{3.23}$$

[6] We ignore the possible existence of bound states of $H$ of which there are usually none; certainly not for the $\pi^+\pi^-$ system of $\pi^+\pi^- \to \rho^0 \to \pi^+\pi^-$.

[7] $L(\hat{p})$ is the rotation free boost.

If there are no degeneracy quantum numbers, i.e., if we ignore the particle species label and channel numbers and restrict ourselves to the case without spins (like for the $\pi^+\pi^-$ system), then the reduced matrix element can be written as

$$\langle\, \eta \,\|\, S_j(\mathsf{s}) \,\|\, \eta' \,\rangle = S_j(\mathsf{s}) \tag{3.24}$$

where $j$ is the orbital angular momentum in the center of mass. We insert (3.23) and (3.24) into (3.18) to obtain for the $S$-matrix element

$$(\psi^-, \phi^+) = \sum_j \int_{(m_1+m_2)^2}^{\infty} d\mathsf{s} \sum_{j_3} \int \frac{d^3\hat{p}}{2\hat{E}(\hat{p})} \langle\, \psi^- \,|\, U(L(\hat{p})) \,|\, \mathbf{0} j_3 j \mathsf{s}^- \rangle \\ S_j(\mathsf{s}) \langle^+ \mathbf{0} j_3 j \mathsf{s} \,|\, U^\dagger(L(\hat{p})) \,|\, \phi^+ \rangle \tag{3.25}$$

Since resonances come in one partial wave with definite value of angular momentum, we consider only the one term in the sum over $j$ with $j = j_R (= 1^-$ for $\pi^+\pi^- \to \rho^0 \to \pi^+\pi^-)$, i.e., we restrict ourselves to the subspace with $j = j_R\, (s = 0,\, l = j,\, n = n_\rho,\, n_{\pi\pi})$. This means that we consider only the term with $j = j_R$ in the sum on the right hand side of (3.25) and call $S_{j_R}(\mathsf{s}) = S(\mathsf{s})$. To simplify the equations, we also consider (3.25) first for fixed values of $j_3$ and $\hat{p}$, say $\hat{\boldsymbol{p}} = \mathbf{0}$ and $j_3 = 0$. We define

$$\begin{aligned} \langle \psi^-(\hat{\boldsymbol{p}}) \,|\, j_3, \mathsf{s}^- \rangle &\equiv \langle \psi^- \,|\, U(L(\hat{p})) \,|\, \mathbf{0}\, j_3,\, j = j_R \mathsf{s}^- \rangle \\ &= \overline{\langle^- \mathsf{s}, j_3 \,|\, \psi^-(\hat{\boldsymbol{p}}) \rangle} \in \tilde{\mathcal{S}} \cap \mathcal{H}^2_-|_{\mathbb{R}_{\mathsf{s}_0}} \\ \langle^+ \mathbf{0}\, j_3,\, j_R \mathsf{s} \,|\, U^\dagger(L(p))\, |\phi^+\rangle &\equiv \langle^+ j_3, \mathsf{s} \,|\, \phi^+(\hat{\boldsymbol{p}}) \rangle \in \tilde{\mathcal{S}} \cap \mathcal{H}^2_-|_{\mathbb{R}_{\mathsf{s}_0}} \end{aligned} \tag{3.26}$$

and write (3.25)

$$(\psi^-, \phi^+) = \sum_{j_3} \int \frac{d^3\hat{p}}{2\hat{E}} (\tilde{\psi}^-_{j_3}(\hat{\boldsymbol{p}}), \tilde{\phi}^+_{j_3}(\hat{\boldsymbol{p}})) \tag{3.27}$$

where

$$\begin{aligned} \left(\tilde{\psi}^-_{j_3}(\hat{\boldsymbol{p}}), \tilde{\phi}^+_{j_3}(\hat{\boldsymbol{p}})\right) &\equiv \int d\mathsf{s} \langle \psi^- \,|\, U(L(\hat{p})) \,|\, \mathbf{0} j_3,\, j_R \mathsf{s}^- \rangle S(\mathsf{s}) \\ &\qquad \langle^+ \mathbf{0} j_3,\, j_R \mathsf{s} \,|\, U^\dagger(L(\hat{p})) \,|\, \phi^+ \rangle \\ &= \int d\mathsf{s} \langle \psi^-(\hat{\boldsymbol{p}}) \,|\, j_3 \mathsf{s}^- \rangle S(\mathsf{s}) \langle^+ j_3 \mathsf{s} \,|\, \phi^+(\hat{\boldsymbol{p}}) \rangle\,. \end{aligned} \tag{3.28}$$

For the fixed values $\hat{\boldsymbol{p}} = \mathbf{0}$, $j_3 = 0$ we suppress the labels $\hat{\boldsymbol{p}}$ and $j_3$ :

$$\left(\tilde{\psi}^-, \tilde{\phi}^+\right) \equiv \left(\tilde{\psi}^-_{j_3=0}(\hat{\boldsymbol{p}} = \mathbf{0}), \tilde{\phi}^+_{j_3=0}(\hat{\boldsymbol{p}} = \mathbf{0})\right) \tag{3.29}$$

and write

$$\left(\tilde{\psi}^-, \tilde{\phi}^+\right) = \int_{(m_1+m_2)^2}^{\infty} d\mathsf{s} \langle \psi^- \,|\, \mathsf{s}^- \rangle S(\mathsf{s}) \langle {}^+\mathsf{s} \,|\, \phi^+ \rangle \,. \tag{3.30}$$

The quantities $(\tilde{\psi}^-, \tilde{\phi}^+)$ are not matrix elements but matrix element densities in the sense that they have to be calculated for all 4-velocities $\hat{p}$ and all $j_3$ and then integrated and summed using (3.27). Considered as functions of the invariant mass-squared $\mathsf{s}$, we make the assumption (3.11), i.e., we assume that the wave functions fulfill

$$\begin{aligned} &\langle {}^+\mathsf{s} \,|\, \phi^+ \rangle \in \tilde{\mathcal{S}} \cap \mathcal{H}^2_-|_{\mathbb{R}_{\mathsf{s}_0}} \\ &\langle {}^-\mathsf{s} \,|\, \psi^- \rangle = \overline{\langle \psi^- \,|\, \mathsf{s}^- \rangle} \in \tilde{\mathcal{S}} \cap \mathcal{H}^2_+|_{\mathbb{R}_{\mathsf{s}_0}} \,, \text{ i.e., } \langle \psi^- \,|\, \mathsf{s}^- \rangle \in \tilde{\mathcal{S}} \cap \mathcal{H}^2_-|_{\mathbb{R}_{\mathsf{s}_0}} \,. \end{aligned} \tag{3.31}$$

Then (3.30) is very much the same as the corresponding expression for the $S$-matrix element in the non-relativistic case (e.g., equation (3.9) in [22] or equation (5.4) in [2]) except that in place of the non-relativistic energy $E$ with lower bound 0 we have here the center of mass energy squared $\mathsf{s} = E^2_{cm}$, with lower bound $(m_1 + m_2)^2$. The reason for this choice is that $\mathsf{s}$ is the variable of the relativistic partial $S$-matrix $S_j(\mathsf{s})$ which is continued to complex values, and the Riemann energy surface of $S_j(\mathsf{s})$ has features similar to the Riemann surface of non-relativistic partial $S$-matrix $S_l(E)$. In particular, in the relativistic $S$-matrix theory $S_j(\mathsf{s})$ is analytic on the first "physical" sheet except for cuts along the real line (and bound state poles if any exist), and resonances are defined by (pairs of) poles at the complex values $\mathsf{s} = \mathsf{s}_R = \left(M_R \pm \frac{i}{2}\Gamma_R\right)^2$ in the second sheet or another "unphysical" sheet.

# 4 Analytic Properties of $S(\mathsf{s})$

We start with the hypothesis that unstable particles and resonances are associated with poles of the relativistic partial $S$–matrix $\langle \eta \,\|\, S_j(\mathsf{s}) \,\|\, \eta' \rangle \equiv S(\mathsf{s})$ on the "unphysical" Riemann sheet. The partial $S$-matrix $S(\mathsf{s})$ for physical values of the center of mass energy $\mathsf{s} \geq (m_1 + m_2)^2$ is the boundary value to the

real axis $\mathsf{s}+i\epsilon$, $\epsilon \to 0^+$ of a function in the complex $\mathsf{s}$-plane which is analytic except for cuts along the real axis and possibly bound state (stable particle) poles $P_i$ at $\mathsf{s} = m_{P_i}^2 < (m_1+m_2)^2$. The right hand cut (a consequence of unitarity) starts at the elastic scattering threshold $\mathsf{s}_{min} = (m_1+m_2)^2$. This cut is two-sheeted [18], i.e., of the square-root type. The elastic scattering matrix element, $\eta' = \eta$, has further branch points at each energy $\mathsf{s}_{th}$ corresponding to a threshold for a newly allowed physical process (e.g., possibly for $\pi^+\pi^- \to \rho \to \pi^+\pi^-\pi^+\pi^-$ at $\mathsf{s}_{th} = (2m_1+2m_2)^2$). To make $S(\mathsf{s})$ single-valued on a Riemann surface, cuts start at these branch-points and are drawn along the real axis, cf. Figure 1. These branch-points are called normal thresholds and start at energies at which production of other particles (inelastic processes) is possible. If one does not cross a cut, one stays on the "physical sheet". The first normal threshold $\mathsf{s} = (m_1+m_2)^2$ is the least $\mathsf{s}$ at which a two particle state can exist. To reach other sheets of the Riemann surface of $S(\mathsf{s})$ one burrows through one or several branch cuts. These sheets are called the "unphysical" sheets. The second sheet is reached from the physical value $\mathsf{s}+i\epsilon$, $\mathsf{s} > (m_1+m_2)^2$ by burrowing down through the normal threshold between $\mathsf{s}_{th1} = (m_1+m_2)^2$ and $\mathsf{s}_{th2}$. The resonance poles $P$ of the relativistic elastic scattering $S$-matrix element is located on the second sheet at $\mathsf{s}_R = (M_R - \frac{i}{2}\Gamma_R)^2$. This hypothesis is suggested by the idea that an unstable particle may be connected to a stable particle by letting the parameters that measure the strength of the force (between the decay products) vary continuously until that force becomes so strong that the unstable particle becomes a stable particle, since a stable particle corresponds to a pole on the physical sheet at the real value $\mathsf{s} = m_{P_i}^2$ below the threshold $(m_1+m_2)^2$; in the transition to instability the pole must pass round the $(m_1+m_2)^2$ branch point and through the cut. Hermitian analyticity (symmetry relation of the $S$–matrix $S(\mathsf{s}-i\epsilon) = S^*(\mathsf{s}+i\epsilon)$) implies that when the pole position is complex there must be a pole $P'$ at the complex conjugate position $\mathsf{s}_R^* = (M_R + \frac{i}{2}\Gamma_R)^2$ on the unphysical sheet reached by burrowing through the cut from the lower half plane of the physical sheet of Figure 1. Thus a scattering resonance is defined by a pair of poles on the second sheet of the analytically continued $S$-matrix located at positions that are complex conjugates of each other. The pole $P'$ corresponds to the time-reversed situation which we do not want to discuss here. (There may exist other resonance poles located at the same or other physical sheets, but we will mainly be concerned here with one pair of poles in the second sheet). There may also be higher order poles [24] or branch-points on the unphysical sheet but we do not venture here into these

complications; for the sake of simplicity, we shall restrict ourselves to the elastic process $\pi^+\pi^- \to \pi^+\pi^-$. and assume further that there are no other resonance poles in the same channel (ignoring a possible $\rho(1450)$).

# 5 Analytic Extension to the Resonance Pole

We want to analytically continue the matrix element density $(\tilde{\psi}^-_{j_3}(\hat{\boldsymbol{p}}), \tilde{\phi}^+_{j_3}(\hat{\boldsymbol{p}}))$ (3.28) so as to encompass the resonance pole contribution to the scattering amplitude. As a specific example, we consider resonance formation in elastic scattering, e.g.,

$$\pi^+\pi^- \to \rho^0 \to \pi^+\pi^- \,. \tag{5.1}$$

We will split (see (5.8) below) $(\tilde{\psi}^-_{j_3}(\hat{\boldsymbol{p}}), \tilde{\phi}^+_{j_3}(\hat{\boldsymbol{p}}))$ into a resonance term and a background term. The resonance term yields the state vector description of the unstable particle, which in this case is the $\rho^0$-meson. The procedure depicted below can be generalized to processes other than (5.1), with the background term being determined by the system of branch cuts specific to the particular process considered.

For (5.1), $\phi^+$ is determined from $\phi^{in}$ (3.3), the incoming two pion state prepared by the preparation apparatus, and $\psi^-$ is determined from $\psi^{out}$ (3.5), the registered two pion state. Since the $\rho$ meson has spin 1, in (3.28) we have $j_R = j_\rho = 1$. The path of integration in (3.28) extends along the positive real axis, just above the normal threshold cuts mentioned in Section 4 from $4m^2$ to $\infty$, where $m$ is the mass of $\pi^+$. As explained in Section 4, if we assume that the least energetic channel following $\pi^+\pi^- \to \pi^+\pi^-$ is $\pi^+\pi^- \to \pi^+\pi^-\pi^+\pi^-$, then $S(\mathsf{s})$ has a square root branch cut between $4m^2$ and $16m^2$. Thus the physical sheet is connected to the second Riemann sheet between $4m^2$ and $16m^2$ and $S(\mathsf{s})$ can be analytically continued everywhere in the second Riemann sheet [18], except for the resonance pole of the $\rho^0$ meson. So, to analytically extend (3.28) we deform the contour of integration between $4m^2$ and $16m^2$ in the first sheet into an infinite semi-circle in the lower half plane of the second sheet (cf. Figure 2), taking into account the $\rho$-pole of $S(\mathsf{s})$. This is the only singularity of the integrand in (3.28) since the wave functions $\langle \psi^-(\hat{\boldsymbol{p}}) \,|\, j_3\mathsf{s}^- \rangle$ and $\langle {}^+j_3\mathsf{s} \,|\, \phi^+(\hat{\boldsymbol{p}}) \rangle$, according to (3.11), are Hardy class from below, hence analytic for $\mathrm{Im}\,\mathsf{s} < 0$. Explicitly, the contour

deformation of (3.28) yields, according to Figure 2,

$$
\begin{aligned}
(\tilde{\psi}^-_{j_3}(\hat{\boldsymbol{p}}), \tilde{\phi}^+_{j_3}(\hat{\boldsymbol{p}})) &= \int_{4m^2}^{\infty} d\mathsf{s} \langle \psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle S(\mathsf{s}) \langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \\
&= \Big[ \int_{4m^2}^{-\infty} d\mathsf{s} \langle \psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle S^{II}(\mathsf{s}-i\epsilon) \langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \\
&\quad + \int_{\mathcal{C}_\infty} d\mathsf{s} \langle \psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle S^{II}(\mathsf{s}) \langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \\
&\quad + \oint d\mathsf{s} \langle \psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle S^{II}(\mathsf{s}) \langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \\
&\quad + \int_{\infty}^{16m^2} d\mathsf{s} \langle \psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle S^{II}(\mathsf{s}-i\epsilon) \langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \Big] \\
&\quad + \int_{16m^2}^{\infty} d\mathsf{s} \langle \psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle S(\mathsf{s}+i\epsilon) \langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \, . \qquad (5.2)
\end{aligned}
$$

In the above expression, the terms between the brackets [ ] result from the analytic continuation of $\int_{4m^2}^{16m^2} d\mathsf{s} \cdots$ into the second Riemann sheet through the square root cut between $4m^2$ and $16m^2$, $S^{II}$ is the $S$-matrix in the second sheet, $\mathcal{C}_\infty$ refers to the infinite semi-circle in the lower half-plane of the second sheet, and the integral $\oint \cdots$ is clockwise around the resonance pole $\mathsf{s}_R = (M_R - i\Gamma_R/2)^2$ ($\mathsf{s}_\rho = \mathsf{s}_R$) in the second sheet. For the analytic extension (5.2) to be meaningful, the integral around $\mathcal{C}_\infty$ should vanish, and the wave functions, which are determined for the physical values $\mathbb{R}_{\mathsf{s}_0} = [4m^2, \infty)$, should have a unique extension to the non-physical values of $\mathsf{s}$, i.e., on $(-\infty, 4m^2)$, which, as can be seen in (5.2) are always occurring in the second sheet. Since our work follows closely the derivation of the non-relativistic Gamow vectors, to justify (5.2), we invoke below the same arguments used for the non-relativistic case.

To prove the vanishing of the $\mathcal{C}_\infty$ integral in (5.2), we make the same assumption about the growth of the $S$-matrix as made in [25], namely the $S$-matrix on the second sheet is bounded by a polynomial, i.e., for large $\mathsf{s}$, there exists a polynomial $P(\mathsf{s})$ such that $|S_{II}(\mathsf{s})| \leq |P(\mathsf{s})|$. With this assumption, we obtain the relation

$$
\begin{aligned}
&\int_{\mathcal{C}_\infty} \left| d\mathsf{s} \langle \psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle S(\mathsf{s}) \langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \right| \\
&\qquad \leq \int_{\mathcal{C}_\infty} \left| d\mathsf{s} \langle \psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle P(\mathsf{s}) \langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \right| \, . \qquad (5.3)
\end{aligned}
$$

From Proposition 3.2, it follows that $P(\mathsf{s})\langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \in \tilde{\mathcal{S}} \cap \mathcal{H}^2_-$. Hence, a straightforward application of Hölder's inequality shows that

$$\langle\psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle P(\mathsf{s})\langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \in \mathcal{H}^1_- \,. \tag{5.4}$$

With (5.3) and (5.4), the vanishing of the integral around $\mathcal{C}_\infty$ follows from Corollary B.1.

As mentioned above, the wave functions $\langle\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^-|\psi^-\rangle$ and $\langle\hat{\boldsymbol{p}}j_3[\mathsf{s}j]^+|\phi^+\rangle$ should be obtained unambiguously everywhere on the real line of the second sheet. The Hardy class assumptions (3.11) ensure that this is actually the case. This follows from the remarkable property that every Hardy class function is completely determined from its values on a half-axis of the real line. In other words, there exists a bijective mapping

$$\theta \;:\; \tilde{\mathcal{S}} \cap \mathcal{H}^2_\mp \rightarrow\; \tilde{\mathcal{S}} \cap \mathcal{H}^2_\mp|_{\mathbb{R}_{\mathsf{s}_0}} \,. \tag{5.5}$$

This result, which follows from a theorem of van-Winter's theorem (B.2, Appendix B below) is as crucial here as it is for the formulation of the non-relativistic Gamow vectors. The $\theta$ function in (5.5) allows the values of the wave functions on the non-physical region of the Mandelstam variable $\mathsf{s}$ on the second sheet to be uniquely determined from their values on the physical range $[4m^2, \infty)$.

Having elucidated how the Hardy class assumptions (3.11) provide the mathematical justification of (5.2), we now consider the integral around the pole separately

$$\oint d\mathsf{s}\langle\psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle S^{II}(\mathsf{s})\langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle \,.$$

For this integral, we expand the partial $S$-matrix $S^{II}(\mathsf{s})$ into a Laurent series about the resonance pole

$$S^{II}(\mathsf{s}) = \frac{r}{\mathsf{s}-\mathsf{s}_R} + A(\mathsf{s}) \,, \tag{5.6}$$

where $r$ is the residue of $S(\mathsf{s})$ at the pole and $A(\mathsf{s})$ is an analytic function. The pole term in (5.2) can then be expressed as

$$\begin{aligned}
&\oint d\mathsf{s}\langle\psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle\langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle\frac{r}{\mathsf{s}-\mathsf{s}_R} \\
&\quad = -2\pi i r\langle\psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}_R^-\rangle\langle^+ j_3\mathsf{s}_R|\phi^+(\hat{\boldsymbol{p}})\rangle \\
&\quad = \int_{-\infty II}^{\infty} d\mathsf{s}\langle\psi^-(\hat{\boldsymbol{p}})|j_3\mathsf{s}^-\rangle\langle^+ j_3\mathsf{s}|\phi^+(\hat{\boldsymbol{p}})\rangle\frac{r}{\mathsf{s}-\mathsf{s}_R} \,.
\end{aligned} \tag{5.7}$$

The first equality in (5.7) the well known theorem of Cauchy and the second equality follows from a theorem of Titchmarsh ((B.1), Appendix B). With (5.7), (5.2) becomes

$$\left(\tilde{\psi}^-_{j_3}(\hat{\boldsymbol{p}}),\tilde{\phi}^+_{j_3}(\hat{\boldsymbol{p}})\right) = -2\pi i\, r\,\langle\psi^-(\hat{\boldsymbol{p}})\,|\,j_3\mathsf{s}_R^-\rangle\langle^+j_3\mathsf{s}_R\,|\,\phi^+(\hat{\boldsymbol{p}})\rangle + B_{j_3}(\hat{\boldsymbol{p}}) \quad (5.8)$$

where $B_{j_3}(\hat{\boldsymbol{p}})$ is given by the other non-zero integrals in (5.2):

$$\begin{aligned} B_{j_3}(\hat{\boldsymbol{p}}) &= \int_{16m^2}^{\infty} d\mathsf{s}\langle\psi^-(\hat{\boldsymbol{p}})\,|\,j_3\mathsf{s}^-\rangle\langle^+j_3\mathsf{s}\,|\,\phi^+(\hat{\boldsymbol{p}})\rangle S(\mathsf{s}+i\epsilon)\\ &\quad -\int_{16m^2}^{\infty} d\mathsf{s}\langle\psi^-(\hat{\boldsymbol{p}})\,|\,j_3\mathsf{s}^-\rangle\langle^+j_3\mathsf{s}\,|\,\phi^+(\hat{\boldsymbol{p}})\rangle S^{II}(\mathsf{s}-i\epsilon)\\ &\quad -\int_{-\infty}^{4m^2} d\mathsf{s}\langle\psi^-(\hat{\boldsymbol{p}})\,|\,j_3\mathsf{s}^-\rangle\langle^+j_3\mathsf{s}\,|\,\phi^+(\hat{\boldsymbol{p}})\rangle S^{II}(\mathsf{s}-i\epsilon)\,. \end{aligned} \quad (5.9)$$

Since (5.8) is valid for any registered two pion state $\psi^-_{\pi^+\pi^-}$, we can omit the arbitrary $\psi^-_{\pi^+\pi^-}\in\Phi_+$ or $\tilde{\psi}^-$ and represent a two pion in-state density by the functional equation in the space $\Phi_+^\times$:

$$\tilde{\phi}^+_{j_3}(\hat{\boldsymbol{p}}) = -2\pi i\, r\,|\hat{\boldsymbol{p}}j_3[\mathsf{s}_R j_R]^-\rangle\langle^+j_3\mathsf{s}_R\,|\,\phi^+(\hat{\boldsymbol{p}})\rangle + |B_{j_3}(\hat{\boldsymbol{p}})\rangle\,, \quad (5.10)$$

where

$$\begin{aligned} |B_{j_3}(\hat{\boldsymbol{p}})\rangle &= \int_{16m^2}^{\infty} d\mathsf{s}|\hat{\boldsymbol{p}}j_3[\mathsf{s}j_R]^-\rangle\langle^+j_3\mathsf{s}\,|\,\phi^+(\hat{\boldsymbol{p}})\rangle S(\mathsf{s}+i\epsilon)\\ &\quad -\int_{16m^2}^{\infty} d\mathsf{s}|\hat{\boldsymbol{p}}j_3[\mathsf{s}j_R]^-\rangle\langle^+j_3\mathsf{s}\,|\,\phi^+(\hat{\boldsymbol{p}})\rangle S^{II}(\mathsf{s}-i\epsilon)\\ &\quad -\int_{-\infty}^{4m^2} d\mathsf{s}|\hat{\boldsymbol{p}}j_3[\mathsf{s}j_R]^-\rangle\langle^+j_3\mathsf{s}\,|\,\phi^+(\hat{\boldsymbol{p}})\rangle S^{II}(\mathsf{s}-i\epsilon)\,. \end{aligned} \quad (5.11)$$

The expansion (5.10) is the relativistic version of the complex basis expansion obtained for the non relativistic case (cf. equation (5.40) of [2]). This is the complex basis expansion if there is one resonance in the partial wave. If there are two or $N$ resonances with $j=j_R$ at $\mathsf{s}=\mathsf{s}_{R_1},\,\mathsf{s}_{R_2},\,\cdots$ (e.g., $\rho(1450)$ and $\rho(770)$ in $\pi^+\pi^-\to\pi^+\pi^-$), then we obtain for each an additional integral around the pole $\mathsf{s}_{R_i}$ in (5.2). For each of these poles separately we follow the above procedure and obtain, in place of the first term on the right hand side of (5.10) and (5.12) below, a sum over the $\mathsf{s}_{R_i}$ (superposition of interfering resonances).

As prescribed in (3.27), the in-state $\phi^+$ of the two pions (in the subspace $j = j_R$) is obtained from the in-state density $\tilde{\phi}^+_{j_3}(\hat{\boldsymbol{p}})$ by integration over the 4-velocities and summing over $j_3$. Thus,

$$\begin{aligned}\phi^+ &= \sum_{j_3} \int \frac{d^3\hat{p}}{2\hat{p}^0} \tilde{\phi}^+_{j_3}(\hat{\boldsymbol{p}}) \\ &= \sum_{j_3} \int \frac{d^3\hat{p}}{2\hat{p}^0} |\hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle \phi_{j_3}(\hat{\boldsymbol{p}}) + |B\rangle \, , \end{aligned} \tag{5.12}$$

where

$$\phi_{j_3}(\hat{\boldsymbol{p}}) \equiv -2\pi i r \langle^+ j_3 \mathsf{s}_R | \phi^+(\hat{\boldsymbol{p}})\rangle = -2\pi i r \langle^+ \hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R] | \phi^+\rangle \, , \tag{5.13}$$

and

$$|B\rangle = \sum_{j_3} \int \frac{d^3\hat{p}}{2\hat{p}^0} |B_{j_3}(\hat{\boldsymbol{p}})\rangle \, . \tag{5.14}$$

The state vector $|\hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle$ which appears in (5.10) is the relativistic Gamow vector that we set out to construct. As apparent from the derivation leading to (5.10), it is obtained from the analytic extension in $\mathsf{s}$ of the Dirac-Lippmann-Schwinger kets $|\hat{\boldsymbol{p}} j_3 [\mathsf{s} j_R]^-\rangle$ to the resonance pole in the second sheet of the analytically continued $S$-matrix. The first term in the right hand side of (5.12), which is a continuous linear superposition of $|\hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle$ over the 4-velocity $\hat{p}$ with fixed values of $\mathsf{s}_R$ and $j_R$, represents a velocity wave packet of the resonance particle. We denote it by

$$\phi^G_{j_R \mathsf{s}_R} = \sum_{j_3} \int \frac{d^3\hat{p}}{2\hat{p}^0} |\hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle \phi_{j_3}(\hat{\boldsymbol{p}}) \, . \tag{5.15}$$

We recall that $\phi_{j_3}(\hat{\boldsymbol{p}})$ in (5.15), defined by (5.13), is a Schwartz function with respect to $\hat{\boldsymbol{p}}$, as required by (3.11b).

To obtain the invariant mass-square distribution of $|\hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle$, we deduce from the second equality in (5.7) that

$$\begin{aligned}\langle \psi^-(\hat{\boldsymbol{p}}) | j_3 \mathsf{s}_R^-\rangle &= \int_{-\infty_{II}}^{\infty} d\mathsf{s} \frac{\langle^+ j_3 \mathsf{s} | \phi^+(\hat{\boldsymbol{p}})\rangle}{\langle^+ j_3 \mathsf{s}_R | \phi^+(\hat{\boldsymbol{p}})\rangle} \frac{\langle \psi^-(\hat{\boldsymbol{p}}) | j_3 \mathsf{s}^-\rangle}{\mathsf{s} - \mathsf{s}_R} \\ &= \int_{-\infty_{II}}^{\infty} d\mathsf{s} \frac{\langle \psi^-(\hat{\boldsymbol{p}}) | j_3 \mathsf{s}^-\rangle}{\mathsf{s} - \mathsf{s}_R} \, , \end{aligned} \tag{5.16}$$

where $-\infty_{II}$ signifies that the "unphysical" values of $\mathsf{s}$, $(-\infty, 4m^2)$, occur in the second sheet. Since (5.16) is valid for any registered two pion state $\psi^-_{\pi^+\pi^-}$, we can represent the state vector corresponding to the resonance particle by

$$|\hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle = \frac{i}{2\pi}\int_{-\infty_{II}}^{\infty} d\mathsf{s} \frac{|\hat{\boldsymbol{p}} j_3 [\mathsf{s} j_R]^-\rangle}{\mathsf{s}-\mathsf{s}_R}\,. \tag{5.17}$$

The integral representation in (5.17) shows that the relativistic Gamow vector has a Breit-Wigner distribution in the $\mathsf{s}$-variable (invariant mass-squared)

$$\langle \hat{\boldsymbol{p}} j_3 [\mathsf{s} j_R]^- | \hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle \sim \frac{1}{\mathsf{s}-\mathsf{s}_R} \qquad -\infty_{II} < \mathsf{s} < \infty\,. \tag{5.18}$$

Exactly as in the non-relativistic case (cf., Proposition 4 in [25]), it can be shown that $|\hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle$ with the integral representation (5.17) is a continuous antilinear functional on $\Phi_+$, i.e., $|\hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle \in \Phi_+^\times$.

The relativistic Gamow vector $|\hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle$ is a generalized eigenvector of $P^\mu$ with a complex eigenvalue. To see this, we use the integral representation (5.7) with the vector $P_\mu \psi^- \in \Phi_+$:

$$\begin{aligned}
\langle P_\mu \psi^- \,|\, \hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle &= \frac{i}{2\pi}\int_{-\infty}^{\infty} d\mathsf{s} \frac{\langle P_\mu \psi^- \,|\, \hat{\boldsymbol{p}} j_3 [\mathsf{s} j_R]^-\rangle}{\mathsf{s}-\mathsf{s}_R} \\
&= \frac{i}{2\pi}\int_{-\infty}^{\infty} d\mathsf{s} \frac{\sqrt{\mathsf{s}}\hat{p}_\mu \langle \psi^- \,|\, \hat{\boldsymbol{p}} j_3 [\mathsf{s} j_R]^-\rangle}{\mathsf{s}-\mathsf{s}_R} \\
&= \sqrt{\mathsf{s}_R}\hat{p}_\mu \langle \psi^- \,|\, \hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle\,.
\end{aligned} \tag{5.19}$$

In (5.19), we used (3.13a) to write $\langle P_\mu \psi^- | \hat{\boldsymbol{p}} j_3 [\mathsf{s} j]^-\rangle = \sqrt{\mathsf{s}}\hat{p}_\mu \langle \psi^- | \hat{\boldsymbol{p}} j_3 [\mathsf{s} j]^-\rangle$ and (3.14) to assert that $\sqrt{\mathsf{s}}\hat{p}_\mu \langle \psi^- | \hat{\boldsymbol{p}} j_3 [\mathsf{s} j]^-\rangle$ is a Hardy class function from below, so that Titchmarsh theorem B.1 can be applied to obtain the last equality. Similarly,

$$\begin{aligned}
\langle M^2 \psi^- \,|\, \hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle &= \frac{i}{2\pi}\int_{-\infty}^{\infty} d\mathsf{s} \frac{\langle M^2 \psi^- \,|\, \hat{\boldsymbol{p}} j_3 [\mathsf{s} j_R]^-\rangle}{\mathsf{s}-\mathsf{s}_R} \\
&= \frac{i}{2\pi}\int_{-\infty}^{\infty} d\mathsf{s} \frac{\mathsf{s}\langle \psi^- \,|\, \hat{\boldsymbol{p}} j_3 [\mathsf{s} j_R]^-\rangle}{\mathsf{s}-\mathsf{s}_R} \\
&= \mathsf{s}_R \langle \psi^- \,|\, \hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle\,.
\end{aligned} \tag{5.20}$$

Equation (5.20), valid for all $\psi^- \in \Phi_+$, is the mathematical expression that $|\hat{\boldsymbol{p}} j_3 [\mathsf{s}_R j_R]^-\rangle$ is a generalized eigenvector of $M^2$ with the complex mass square

$\mathsf{s}_R$ as eigenvalue. This is written equivalently as

$$M^{2\times}|\hat{\boldsymbol{p}}j_3[\mathsf{s}_R j_R]^-\rangle = \mathsf{s}_R|\hat{\boldsymbol{p}}j_3[\mathsf{s}_R j_R]^-\rangle\,. \tag{5.21}$$

In exactly the same way, the wave-packet (5.15) is a generalized eigenvector of the mass square operator $M^2$ with the eigenvalue $\mathsf{s}_R$:

$$M^{2\times}\phi^G_{j_R\mathsf{s}_R} = \mathsf{s}_R\phi^G_{j_R\mathsf{s}_R}\,. \tag{5.22}$$

Hence, $\phi^G_{j_R\mathsf{s}_R}$ in (5.15) represents velocity wave-packets of the unstable particles associated with the pole $\mathsf{s}_R$.

# 6 Conclusion

In this paper we have discussed a state-vector description of relativistic unstable particles. Following the norms of particle physics phenomenology, we can anticipate at the outset that the states of an unstable particle ought to be characterized by the values of its mass, width and spin, the first two of which have been combined in several different ways to a single complex quantity which is associated with the position of the resonance pole $\mathsf{s}_R$ of the relativistic $S$-matrix. As affirmed by Wigner's classic paper [8], a stable particle can be given a state vector description where the state vectors are specified by the generalized eigenvalues of momenta, real mass and spin. These state vectors in fact furnish an irreducible unitary representation of the Poincaré group.

In the theory we have proposed in this paper, the state vector description of unstable particles arises from the relativistic Gamow vector $|\hat{\boldsymbol{p}}j_3[\mathsf{s}_R j_R]^-\rangle$, which are vectors (associated to and) obtained from the pole term of the $S$-matrix at $\mathsf{s} = \mathsf{s}_R$. It is a simultaneous *generalized* eigenvector of the invariant operators $M^2 = P_\mu P^\mu$ and $\hat{W} = -\hat{w}_\mu \hat{w}^\mu$ [8] as well as the four velocity operators $\hat{P}_\mu$ and the third spin component $\hat{w}_3$. Evidently, the Gamow vector is a generalized eigenvector of the momenta as well, an immediate consequence of the defining identity $P_\mu = M\hat{P}_\mu = \sqrt{\mathsf{s}_R}\hat{P}_\mu$. Thus there exists a manifest parallel between Wigner's state vector description of stable particles and the Gamow vector educed state vector description of unstable particles. The fundamental difference is the complexness of the (generalized) eigenvalue of the invariant mass square operator $M^2$ in the latter case.

[8] Here $\hat{w} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu J^{\rho\sigma}$.

The relativistic Gamow vector $|\hat{\boldsymbol{p}}j_3[\mathsf{s}_R j_R]^-\rangle$ with the complex generalized eigenvalue $\sqrt{\mathsf{s}_R}$ of the (essentially) self-adjoint mass operator $M$ was derived here by analytically extending the Dirac-Lippmann-Schwinger kets $|\hat{\boldsymbol{p}}j_3[\mathsf{s}j_R]^-\rangle$ in the Mandelstam variable $\mathsf{s}$ into the lower half complex plane from its initial range $(m_1+m_2)^2 \leq \mathsf{s} < \infty$, appertaining to a scattering process of two particles, $(m_1, j_1)$ and $(m_2, j_2)$. The point $\mathsf{s}_R$ in the lower half plane at which this analytic extension yields the Gamow vector $|\hat{\boldsymbol{p}}j_3[\mathsf{s}_R j_R]^-\rangle$ is where there exists a simple pole of the analytic $S$-matrix designating the unstable particle.

Aside from being an alluring proposition on theoretical grounds, for it puts our understanding of stable and unstable particles on an equal footing, this state vector description of unstable particles by Gamow vectors can be seen to unify various properties heuristically attributed to unstable particles and resonances. First, Eqs. (5.16) and (5.17) show that the Gamow vector $|\hat{\boldsymbol{p}}j_3[\mathsf{s}_R j_R]^-\rangle$ can be resolved into an integral representation over a Breit-Wigner distribution $\frac{1}{\mathsf{s}-\mathsf{s}_R}$ in the Mandelstam variable on the second sheet, an attribute of its association to the resonance pole at $\mathsf{s}_R$. Although not discussed here, it will be shown in the sequel to this paper that the time evolution of the Gamow vectors is exactly exponential. Further, this exponential time evolution will be shown to be given by $e^{-\Gamma_R t}$ (with $t$ being the time in the rest frame) where $\mathsf{s}_R = (M_R - i\Gamma_R/2)^2$ is the $S$-matrix pole position. This implies that the width of the Breit-Wigner distribution of the particle is related to its lifetime $\tau_R$ via the *exact* identity $\Gamma_R = \frac{\hbar}{\tau_R}$; we can then define $M_R$ as the real mass of the resonance. Thus the characterization of unstable particles proposed here sheds some light on the recent debate among particle physics phenomenologists and theorists on the definition of the mass of the $Z$-boson [10] and certain hadronic resonances [21].

We shall derive the time evolution of Gamow vectors in the forthcoming sequel where we shall study the transformation properties of the Gamow vectors under Lorentz transformations and space-time translations. This study will show that the linear span of Gamow vectors $|\hat{\boldsymbol{p}}j_3[\mathsf{s}_R j_R]^-\rangle$, $\hat{\boldsymbol{p}} \in \mathbb{R}^3$, furnishes an irreducible representation of the causal Poincaré semigroup, defined as a semi-direct product of the group of homogeneous Lorentz transformations with the *semigroup* of space-time translations into the forward light cone. From our requirement that the complexness of the momenta be solely due to that of mass, i.e., by the required reality of the velocities $\hat{p}_\mu = \frac{p_\mu}{\sqrt{\mathsf{s}_R}}$, it will follow that the (homogeneous) Lorentz subgroup can still be represented

unitarily. However, translations will only form a semigroup, not a group. This group theoretical study will further support our view that the characterization of stable and unstable particles need not be qualitatively different in that the particles of either kind can be specified according to their mass and spin by means of irreducible representations of the Poincaré transformations: in the stable case, we have irreducible unitary representations of the Poincaré group, labeled my real masses; and in the unstable case, irreducible representations of the causal Poincaré semigroup, labeled by complex masses. The semigroup of the latter case can be interpreted as representing the asymmetry of time evolution (say, at rest) of unstable systems and the irreversible character of decay processes.


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# Appendices

# A Overview of Rigged Hilbert Space Concepts

A Rigged Hilbert Space [26] is the result of the completion of a scalar product space with respect to three different topologies. The completion of a vector space with respect to some topology $\tau$ amounts to including in this space the limit points of all $\tau$-Cauchy sequences. If one starts with a scalar-product space $\Psi$ and completes it with respect to the norm induced by the scalar product

$$\|\phi\| = \sqrt{(\phi, \phi)}\,,$$

a Hilbert space $\mathcal{H}$ is obtained. On the other hand, if $\Psi$ is completed with respect to a topology $\tau_\Phi$ defined by a countable number of norms with some qualifications, a countably normed space $\Phi$ is obtained. This countably normed topology $\tau_\Phi$ is finer than the Hilbert space topology $\tau_{\mathcal{H}}$ so that there are more $\tau_\Phi$-neighborhoods than $\tau_{\mathcal{H}}$-neighborhoods. Hence:

$$\Psi \subset \Phi \subset \mathcal{H}\,.$$

A third space of interest is the space of antilinear functionals on $\Phi$, denoted by $\Phi^{\times}$. Since $\Phi \subset \mathcal{H}$, it follows that $\mathcal{H}^{\times} \subset \Phi^{\times}$. But, from Hilbert space theory, $\mathcal{H} = \mathcal{H}^{\times}$. Hence

$$\Phi \subset \mathcal{H} \subset \Phi^{\times} \,. \tag{A.1}$$

The triplet (A.1) is called a Rigged Hilbert Space when $\Phi$ is nuclear and dense in $\mathcal{H}$ (with respect to $\tau_{\mathcal{H}}$). The fundamental physical axiom of the Rigged Hilbert Space formulation of quantum physics is that the set of states of the physical system do not inhabit the entire Hilbert space $\mathcal{H}$ but an appropriately defined dense subspace $\Phi$ of $\mathcal{H}$. The countably normed topology of $\Phi$ is constructed so as to yield the algebra of relevant physical observables continuous as mappings on $\Phi$. It is this feature of Rigged Hilbert Space theory that is made use of in Section 3.2 in making the distinction between the set of prepared states $\Phi_-$ and registered observables $\Phi_+$ by taking $\Phi_\pm$ as dense subspaces of the same Hilbert space $\mathcal{H}$ as in (3.6). This distinction is what allows semigroup time evolution to be incorporated into the quantum mechanical theory.

The action of an element $F \in \Phi^{\times}$ on $\phi \in \Phi$, $F(\phi)$, is denoted–in the Dirac bra-ket notation–by

$$F(\phi) = \langle \phi | F \rangle \,.$$

Since $\mathcal{H} \subset \Phi^{\times}$, it follows that the Dirac bra-ket $\langle \phi | F \rangle$ is an extension of the Hilbert space scalar product in the sense that

$$\langle \phi | F \rangle = (\phi, F) \quad \text{for} \quad F \in \mathcal{H} \,.$$

The topology on $\Phi^{\times}$, denoted by $\tau_{\Phi^{\times}}$, is the weak*-topology induced by $\Phi$ on $\Phi^{\times}$. This means that convergence in $\Phi^{\times}$ is defined by

$$F_i \xrightarrow{\tau_{\Phi^{\times}}} F \Longleftrightarrow \langle \phi | F_i \rangle \to \langle \phi | F \rangle \,, \quad \text{for all } \phi \in \Phi \,. \tag{A.2}$$

To every $\tau_{\Phi}$-continuous operator $A$ on $\Phi$, there corresponds a $\tau_{\Phi^{\times}}$-continuous operator $A^{\times}$ defined on $\Phi^{\times}$ by

$$\langle \phi | A^{\times} F \rangle \equiv \langle A\phi | F \rangle \,, \quad \text{for all } \phi \in \Phi \,, \quad F \in \Phi^{\times} \,. \tag{A.3}$$

The operator $A^{\times}$ is called the conjugate operator of $A$. It is an extension of the Hilbert space adjoint operator $A^{\dagger}$, since for $F \in \mathcal{H}$ we have

$$\langle \phi | A^{\times} F \rangle = (A\phi, F) = (\phi, A^{\dagger} F) \quad \text{for } F \in \mathcal{H} \,. \tag{A.4}$$

Hence,

$$A^\dagger|_\Phi \subset A^\dagger \subset A^\times \,. \tag{A.5}$$

It should be stressed that the conjugate operator $A^\times$ can be defined as a $\tau_{\Phi^\times}$-continuous operator only when $A$ is a continuous linear operator on $\Phi$. In quantum mechanics, it is impossible (empirically) to restrict oneself to continuous (and therefore bounded) operators $\bar{A}$ in $\mathcal{H}$. However, one can restrict oneself to algebras of observables $\{A, B, \cdots\}$ described by continuous operators in $\Phi$, if the topology of $\Phi$ is suitably chosen. Then, $A^\times$, $B^\times$, $\cdots$ are defined and continuous in $\Phi^\times$.

A generalized eigenvector $|F\rangle$ of a $\tau_\Phi$-continuous operator $A$ with a generalized eigenvalue $\omega \in \mathbb{C}$ is defined by the relation

$$\langle A\phi|F\rangle = \langle \phi|A^\times F\rangle = \omega\langle\phi|F\rangle \,, \qquad \text{for all } \phi \in \Phi \,. \tag{A.6}$$

Since the vector $\phi$ in (A.6) is arbitrary, (A.6) can be formally expressed as

$$A^\times|F\rangle = \omega|F\rangle \,. \tag{A.7}$$

In the Dirac notation the $^\times$ in (A.7) is suppressed so that (A.7) reads

$$A|F\rangle = \omega|F\rangle \,. \tag{A.8}$$

If $A$ is a self-adjoint operator, suppressing the $^\times$ as in (A.8) does not lead to confusion since $A = A^\dagger \subset \Phi^\times$. However, if $A$ is not self-adjoint, a clear distinction between the operator and its conjugate should be made. The concept of generalized eigenvectors (A.7) in Rigged Hilbert Space mathematics allows the description of "eigenstates" which do not exist in the Hilbert space. For instance, the Dirac scattering kets are generalized eigenvectors with eigenvalues belonging to the continuous spectrum, and they are not Hilbert space elements. The Gamow vectors, which are used to describe decaying states, are also generalized eigenvectors which are not in $\mathcal{H}$, but, unlike in the case of scattering states, their complex eigenvalues do not belong to the Hilbert space spectrum of the Hamiltonian.

# B Hardy Class Functions on a Half-plane

**Definition B.1 ($\mathcal{H}^p_\pm$ $1 \le p < \infty$).** A complex function $f(x+iy)$ analytic in the open lower half complex plane ($y < 0$) is said to be a Hardy [27] class function from below of order $p$, $\mathcal{H}^p_-$, if $f(x+iy)$ is $L^p$-integrable as a function of $x$ for any $y < 0$ and

$$\sup_{y<0} \int_{-\infty}^{\infty} dx \, |f(x+iy)|^p < \infty \, . \tag{B.1a}$$

Similarly, a complex function $f(x+iy)$ analytic in the open upper half complex plane ($y > 0$) is said to be a Hardy class function from above of order $p$, $\mathcal{H}^p_+$, if $f(x+iy)$ is $L^p$-integrable as a function of $x$ for all $y > 0$, and

$$\sup_{y>0} \int_{-\infty}^{\infty} dx \, |f(x+iy)|^p < \infty \, . \tag{B.1b}$$

A property of $\mathcal{H}^p_\pm$ functions is that their boundary values on the real axis exist almost everywhere and define an $L^p$-integrable function, i.e., if $f \in \mathcal{H}^p_\pm$, then its boundary values $f(x) \in L^p(\mathbb{R})$. Conversely, the values of any $\mathcal{H}^p_\pm$ function on the upper/lower half-plane are determined from its boundary values on the real axis. This result is provided by a theorem of Titchmarsh:

**Theorem B.1 (Titchmarsh theorem).** If $f \in \mathcal{H}^p_-$, then

$$f(z) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt \, , \qquad \text{for } \operatorname{Im} z < 0 \, ,$$

and

$$\int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt = 0 \, , \qquad \text{for } \operatorname{Im} z > 0 \, .$$

Similarly, if $f \in \mathcal{H}^p_+$, then

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt \, , \qquad \text{for } \operatorname{Im} z > 0 \, ,$$

and

$$\int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt = 0 \, , \qquad \text{for } \operatorname{Im} z < 0 \, .$$

This one-to-one correspondence between the $\mathcal{H}^p_\pm$ functions and their boundary values on $\mathbb{R}$ allows the identification of $f(z)$ with $f(x)$ for $f \in \mathcal{H}^p_\pm$.

The following results are related to the decay properties of the Hardy class functions. They are straightforward generalizations of the corresponding results of [25] and are needed for the construction of the relativistic Gamow vectors.

**Proposition B.1.** Let $\mathcal{C}_\infty$ be the infinite semi-circle in the lower half complex plane. If $f \in \mathcal{H}^p_-$, then

$$\int_{\mathcal{C}_\infty} \left| \frac{f(z)}{z} dz \right| = 0\,.$$

*Proof.* Let $\mathcal{C}_r$ be the arc with radius $r$ shown in Figure 3. Then

$$\left| \int_{\mathcal{C}_r} \frac{f(z)}{z} dz \right| \leq \int_{\mathcal{C}_r} |f(re^{i\theta})| d\theta = \int_{1/r}^{\pi - 1/r} |f(-re^{i\theta})| d\theta\,.$$

Since $f \in \mathcal{H}^p_-$, then there exists $C$ such that

$$|f(-re^{i\theta})| \leq \frac{C}{(r \sin\theta)^{1/p}}\,, \quad \text{(cf. [28] page 149)}\,.$$

Thus

$$\int_{\mathcal{C}_r} \left| \frac{f(z)}{z} dz \right| \leq \frac{2C}{r^{1/p}} \int_{1/r}^{\pi/2} \frac{1}{(\sin\theta)^{1/p}} d\theta\,. \tag{B.2}$$

Using

$$\sin\theta \geq \theta - \theta^3/6 \geq \theta(1 - \pi^2/24)\,, \text{ for } 1/r \leq \theta \leq \pi/2\,,$$

we obtain for (B.2)

$$\begin{aligned} \int_{\mathcal{C}_r} \left| \frac{f(z)}{z} dz \right| &\leq \frac{2C}{r^{1/p}(1 - \pi^2/24)^{1/p}} \int_{1/r}^{\pi/2} \frac{d\theta}{\theta^{1/p}} \\ &= \frac{2C}{(1 - \pi^2/24)^{1/p} r^{1/p}} \begin{cases} \log\left(\frac{r\pi}{2}\right) & p = 1 \\ \frac{1}{1 - \frac{1}{p}} \left[ \left(\frac{\pi}{2}\right)^{1-1/p} - \left(\frac{1}{r}\right)^{1-1/p} \right] & 1 < p < \infty \end{cases} \end{aligned} \tag{B.3}$$

Therefore, as $r \to \infty$, we obtain

$$\int_{\mathcal{C}_\infty} \left| \frac{f(z)}{z} dz \right| = 0\,.$$

□

**Corollary B.1.** Let $f \in \mathcal{S} \cap \mathcal{H}^2_-$, $g \in \mathcal{S} \cap \mathcal{H}^2_-$, then

$$\int_{\mathcal{C}_\infty} |f(z) g(z) dz| = 0\,.$$

*Proof.* Since $f \in \mathcal{S} \cap \mathcal{H}^2_-$, then $xf(x) \in \mathcal{S} \cap \mathcal{H}^2_-$ [25]. A straightforward application of Hölder's inequality shows that $xf(x)g(x) \in \mathcal{H}^1_-$. Then, from the above lemma

$$\int_{\mathcal{C}_\infty} |f(z)g(z)dz| = \int_{\mathcal{C}_\infty} \left| \frac{zf(z)g(z)}{z} dz \right| = 0\,.$$

$\square$

A remarkable property of Hardy class functions that is used in [25] is that they are uniquely determined from their boundary values on a semi-axis on the real line. This result is provided by a theorem of van Winter [29]. Before stating the van Winter's theorem below, we define first the Mellin transform

**Definition B.2 (Mellin transform).** Let $f(x)$ be a function on $\mathbb{R}_+$. Its Mellin transform is a function defined almost everywhere on $\mathbb{R}$ as

$$H(s) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty f(x) x^{is-1/2} dx\,,$$

provided that the integral exists for almost all $s \in \mathbb{R}$.

**Theorem B.2 (van Winter).** A function $f(x) \in L^2(\mathbb{R}^+)$ can be extended to $\mathbb{R}^- = (-\infty, 0]$ to become a function in $\mathcal{H}^2_+$ if and only if its Mellin transform satisfies

$$\int_{-\infty}^\infty (1 + e^{2\pi s}) |H(s)|^2 ds < \infty\,.$$

This extension is unique. The values of $f(z)$ for $z = \rho e^{i\theta}$ for $0 \le \theta \le \pi$, $\rho > 0$ are given by

$$f(\rho e^{i\theta}) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^\infty H(s) (\rho e^{i\theta})^{-is-1/2} ds\,.$$

In particular for negative values of $x$, $f(x)$ is given by

$$f(-x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^\infty H(s) (x e^{i\pi})^{-is-1/2} ds\,.$$

A similar result can be obtained for $\mathcal{H}^2_-$.

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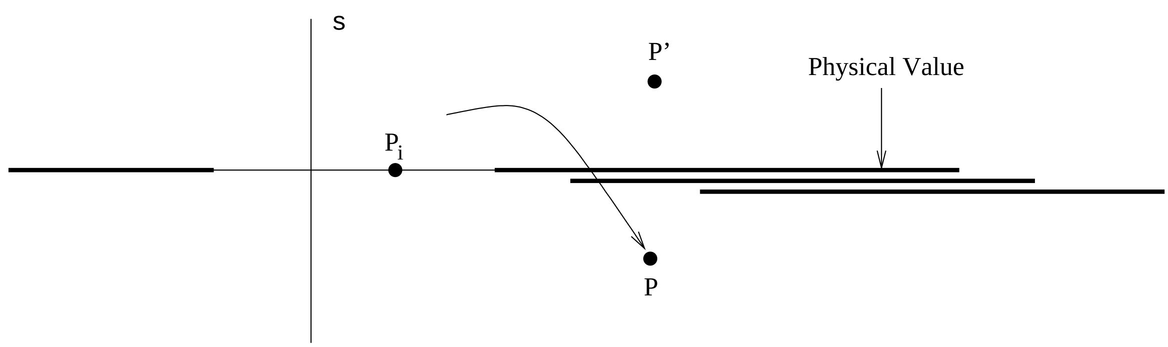


Figure 1: Analytic properties of $S(\mathsf{s})$.

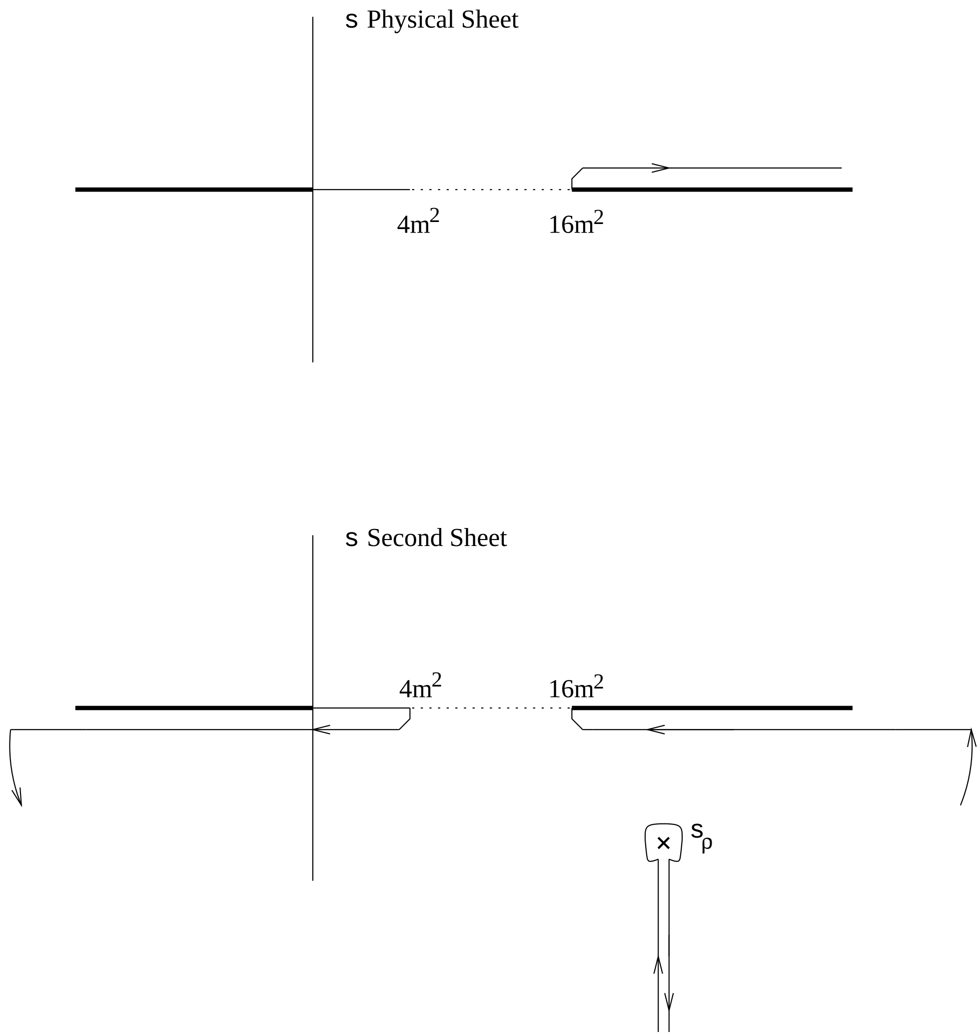


Figure 2: Contour for the analytic continuation of (3.28).

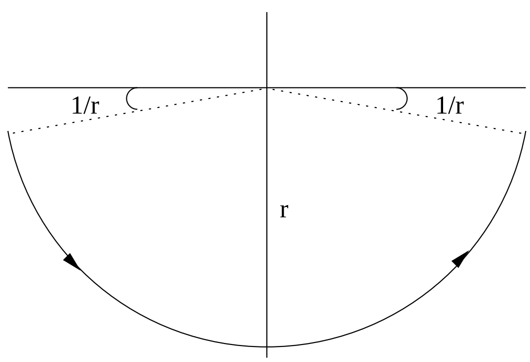


Figure 3: Arc of radius r, $\mathcal{C}_r$.